Uncertainty & Decision

Introduction and basic of representation of preference

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Goals of the project

- Learn the basics of individual decision under uncertainty
- Learn the basics of collective decision
(with complete information)
- Explore the literature on collective decision theory under uncertainty

Structure of the project

- week 1 to 3: lectures
- week 4: Presentation of one article from the literature (June 24th?)
- week 5: write a final paper

Schedule (tentative):

- Introduction + Representation (today)
- von Neumann Morgenstern theorem (Tuesday 7^{th})
- Approach from Savage and De Finetti (Fri 10^{th})
- Arrow's theorem (Mon 13^{th})
- Arrow-Debreu's theorem (Wed 15^{th})
- Overview of aggregation under uncertainty (Fri 17^{th})
- Condorcet Jury Theorem (Tue 21^{st})

References:

- Notes on the theory of Choice. D. Kreps
- Rational choice. I. Gilboa
- Theory of decision under uncertainty. I. Gilboa
- An introduction to Decision Theory. M. Peterson
- Theory of Distributed Justice. J. Roemer
- Theory of value. G. Debreu

Normative and Descriptive Decision Theory

descriptive: building theories that seek to explain and predict how people actually make decision

normative: yield prescription about what decision makers are rationally required or ought to do.

- normative theory can be expected to withstand time and cultural differences
- maybe it is possible to build theories in which beliefs and desires may be aggregated into rational decisions.

A decision is **rational** iff the decision maker chooses to do what she has most reason to do at the point in time which the decision is made (instrumental rationality).

Types of decision

under Complete knowledge under Risk: the decision maker knows the probability of the outcomes.
under ignorance: the probabilities are unknown or nonexistent.
under uncertainty: synonym for ignorance, or broader term covering risk and ignorance.

When there are many decision makers:

- game theory: hypothesis about the opponent
- social choice theory: in most cases, decision under complete information.

Representation

Preference Relation

Let X be a set of outcomes and let $\succ$ be a binary relation to express ones strict preferences over the outcomes.

- **asymmetry**: $x \succ y \Rightarrow y \not\succ x$
- **transitivity**: $(x \succ y \wedge y \succ z) \Rightarrow x \succ z$ *has been questioned*
- **irreflexivity**: $\forall x \in X\ x \not\succ x$
- **negative transitivity**: $x \not\succ y \wedge y \not\succ z \Rightarrow x \not\succ z$ stronger version of transitivity (neg. transitivity implies that indifference is transitive)

Lemma

A binary relation B is negatively transitive iff

$$xBz \Rightarrow (\forall y \in X, xBy \vee yBz)$$

Is it reasonable to ask for neg transitivity?

Definition (Preference relation)

A binary relation $\succ$ is called a **preference relation** if it is **asymmetric** and **negatively transitive**

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Proposition

If $\succ$ is a preference relation, then $\succ$ is irreflexive, transitive and acyclic.

Proposition

We can define weak preference ($\succeq$) and indifference ($\sim$) relations by $x \succeq y$ if $y \not\succ x$ and $x \sim y$ if $x \not\succ y$ and $y \not\succ x$.

Proposition

If $\succ$ is a preference relation, then

- $\forall (x,y) \in X^2$, exactly one of $x \succ y$, $y \succ x$ or $x \sim y$ holds.
- $\succeq$ is complete and transitive.
- $\sim$ is reflexive, symmetric, and transitive.
- $w \succ x$, $x \sim y$, $y \succ z \Rightarrow w \succ y$ and $x \succ z$.
- $x \succ y$ iff $x \succ y$ or $x \sim y$.
- $x \succ y$ and $y \succeq x \Rightarrow x \sim y$.

Proposition

We can define $\succ$ and $\sim$ from a binary relation $\succeq$

$$x \succ y \text{ if } y \not\succeq x$$
$$x \sim y \text{ if } x \succeq y \text{ and } y \succeq x.$$

If $\succeq$ is complete and transitive, then $\succ$ is a preference relation.

Revealed Preference

Assumption: $x \succ y$ iff you choose x over y whenever given the opportunity. ⇒behavior reveals the preference.

This is debatable, perhaps

- You chose by mistake!
- You did not know y was available!
- In this case, it is more difficult to distinguish strict preference with indifference.

Revealed Preference

We consider a choice function c and we generate a binary relation $\succ$ such that

$$x \succ y \text{ iff } x \in c(\{x,y\}).$$

Let $\mathcal{P}(X)$ be the set of non-empty subset of X.

Definition (choice function)

A choice function for a finite set X if a function $c : \mathcal{P}(X) \to \mathcal{P}(X)$ such that $\forall A \subseteq X$, $c(A) \subseteq A$.

We define a function $c(\cdot,\succ)$ of $\mathcal{P}(X)$ by

$$c(A,\succ) = \{x \in A \mid \forall y \in A, y \not\succ x\}.$$

A choice function is **normal** or **rationalizable** iff the binary relation $\succ$ generated from c coincides with $c(\cdot,\succ)$.

Proposition

If a binary relation is **acyclic**,
then $c(\cdot, \succ)$ is **a choice function**.

Proposition

For a binary relation $\succ$,
$c(\cdot, \succ)$ is a choice function iff $\succ$ is acyclic

Definition (Houthakker's axiom)

if $x, y \in A \cap B$ and $x \in c(A)$ and $y \in c(B)$ then $x \in c(B)$

Definition (Sen α – Contraction consistency)

if $x \in B \subseteq A$ and $x \in c(A)$ then $x \in c(B)$.

"If the world champion is a Pakistani, then he must be champion of Pakistan"

Definition (Sen β – Expansion Consistency)

if $x, y \in c(A), A \subseteq B$ and $y \in c(B)$ then $x \in c(B)$.

"If the world champion is a Pakistani, then all champions of Pakistan are also world champions"

Proposition

For an arbitrary binary relation $\succ$, $c(\cdot,\succ)$ satisfies Sen's α property.

Proposition

If $\succ$ is a preference relation, then $c(\cdot,\succ)$ satisfies Houthakker's axiom, hence both Sen's α and β conditions.

Proposition

A choice function c satisfies both properties α and β then there exists a preference relation $\succ$ such that c is $c(\cdot,\succ)$.

Ordinal utility

We are looking for a numerical representation of $\succ$, i.e. a function $u : X \to \mathbb{R}$ such that $x \succ y$ iff $u(x) > u(y)$.

NB:

- **ordinal scales**: better object are assigned higher numbers, the numbers do not reflect any information about differences or ratios between objects.
- **interval scales**: differences have a meaning
- **ratio scales**: ratios have a meaning

Proposition

For X countable, a binary relation $\succ$ is a preference relation iff there exists a function $u : X \to \mathbb{R}$ such that $x \succ y$ iff $u(x) > u(y)$.

If the set is uncountable, there may not exist a function u that represents the preference relation.
E.g. the lexicographic preference relation.

Proposition

For an arbitrary set X and binary relation $\succ$, there exists a function $u : X \to \mathbb{R}$ such that the property ($x \succ y$ iff $u(x) > u(y)$) holds iff $\succ$ is a preference relation and there is a countable $\succ$-order dense subset Z of X.

Uniqueness

Theorem

Given a set X, a preference relation $\succ$ and functions u and u' that represent $\succ$, there exists an increasing function $f : \mathbb{R} \to \mathbb{R}$ such that

- f is strictly increasing
- $u' = f \circ u$

Moreover, for any increasing function $g : \mathbb{R} \to \mathbb{R}$, $g \circ u$ represents $\succ$.

Extensions

Theorem

If X is a subset of a separable metric space, then $\succ$ is a continuous preference relation iff there exists some continuous function $u : X \to \mathbb{R}$ such that $x \succ y$ iff $u(x) > u(y)$.

Next

von Neumann and Morgenstern's theorem