

A clear and robust interpretation of interactions between criteria when constructing multidimensional indexes

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At the beginning...

How to define in a discrete case

- “Criteria i and j **interact**” ?
- “Criteria i and j **are not independent**” ?

We are not talking about correlation (statistics/probabilities)

At the beginning...

An answer given by ChatGPT...



You

define interactions between criteria in multiple criteria decision analysis



ChatGPT

In Multiple Criteria Decision Analysis (MCDA), the interactions between criteria refer to the way in which the performance or evaluation of one criterion can be influenced by the presence or performance of another criterion. Understanding these interactions is crucial for making informed and effective decisions in complex decision-making scenarios. There are several types of interactions that can be considered in MCDA:

1. **Positive Interaction (Synergy):**

- Occurs when the combined effect of two or more criteria is greater than their individual effects.
- A positive change in one criterion enhances the impact or benefits of another criterion.

2. **Negative Interaction (Trade-off or Competition):**

- Occurs when the performance of one criterion negatively affects the performance or value of another criterion.
- A positive change in one criterion may come at the expense of another criterion.

3. **Non-Interaction (Independence):**

- Criteria are considered independent if changes in one criterion do not affect



3. **Non-Interaction (Independence):**

- Criteria are considered independent if changes in one criterion do not affect the evaluation of another criterion.
- The performance of one criterion is not influenced by the presence or absence of other criteria.

4. **Threshold Interaction:**

- Refers to situations where the effect of one criterion becomes significant only when it exceeds a certain threshold value.
- Below the threshold, the impact may be minimal or nonexistent.

5. **Complementary Interaction:**

- Similar to positive interaction, but the improvement in one criterion complements or enhances the effectiveness of another criterion.

6. **Redundancy:**

- Occurs when two or more criteria provide essentially the same information or contribute to the decision in a similar manner.
- Redundant criteria may be simplified or one may be excluded without significantly affecting the decision.

7. **Conflicting Interaction:**

- Different criteria may conflict with each other, making it challenging to optimize all criteria simultaneously.
- Trade-offs or compromises may be necessary to find a balanced solution.



The answer is not obvious and clear !

MultiCriteria Decision Analysis

Aim: to help a Decision-Maker (DM) to select one or more alternatives among several alternatives evaluated on $|N|$ criteria often contradictory.

- We need to construct a preference relation \succsim_X over the set of all alternatives $X = X_1 \times X_2 \times \dots \times X_n$;
- The DM provides a preference information $\{P, I\}$ on a subset $X' \subseteq X$

Preferences and interactions

Example (A classic example of Grabisch et al. (2010))

	1 : Mathematics (M)	2 : Statistics (S)	3 : Language (L)
<i>a</i>	16	13	7
<i>b</i>	16	11	9
<i>c</i>	6	13	7
<i>d</i>	6	11	9

- For a student good in Mathematics, Language is more important than Statistics

$$\implies b \succ_X a,$$

- For a student bad in Mathematics, Statistics is more important than Language

$$\implies d \prec_X c.$$

Preferences and interactions

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- $b \succ_X a$ and $d \prec_X c$ do not satisfy the independence (**mutual preference independence**) axiom since

$$(16, 11, 9) \succ_X (16, 13, 7) \text{ and } (6, 11, 9) \prec_X (6, 13, 7).$$

Preferences and interactions

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- Therefore $b \succ_X a$ and $d \prec_X c$ are not representable by an additive model.
- We try to represent them by a non-additive model: e.g. a 2-additive Choquet integral.

The 2-additive Choquet integral

Definition

For any $x := (x_1, \dots, x_n) \in X$, the expression of the 2-additive Choquet integral is:

$$C_\mu(u_1(x_1), \dots, u_n(x_n)) = \sum_{i=1}^n \phi_i^\mu u_i(x_i) - \frac{1}{2} \sum_{\{i,j\} \subseteq N} l_{ij}^\mu |u_i(x_i) - u_j(x_j)|$$

Where

- l_{ij}^μ = the interaction index between criteria i and j :

$$l_{ij}^\mu = m_{ij}^\mu = \mu_{ij} - \mu_i - \mu_j.$$

- ϕ_i^μ = the importance of the criterion i (\equiv Shapley index):

$$\phi_i^\mu = \mu_i + \frac{1}{2} \sum_{k \in N \setminus i} l_{ik}^\mu.$$

Shapley Interaction index $I_{ij}^\mu = \mu_{ij} - \mu_i - \mu_j$

Usual interpretation

- $I_{ij}^\mu > 0 \implies$ criteria i and j are *complementarity*.
- $I_{ij}^\mu < 0 \implies$ criteria i and j are *redundant*.
- $I_{ij}^\mu = 0 \implies$ criteria i and j are *independent* (no interaction).

The sign of $I_{ij}^{\mu} = \mu_{ij} - \mu_i - \mu_j$ is not always stable

	Par. 1	Par. 2	Par.3	Par. 4	Par. 5	Par. 6	Par. 7	Par.8	Par. 9
$C_{\mu}(a)$	8.5	13.75	9.1	13.765	13.75	13.75	11.47	12.535	10.45
$C_{\mu}(b)$	9.5	14.25	9.7	13.995	14.25	14.25	11.93	12.785	10.75
$C_{\mu}(c)$	7.75	9.75	7.75	11.325	11.25	9.75	9.45	9.515	7.85
$C_{\mu}(d)$	7.25	9.25	7.25	10.295	9.75	9.25	8.91	9.265	7.55
μ_M	0	0.75	0	0.685	0.75	0.75	0.36	0.485	0.15
μ_S	0.25	0.5	0.25	0.73	0.75	0.5	0.465	0.455	0.25
μ_L	0	0.25	0	0.315	0	0	0.205	0.32	0
μ_{MS}	0.25	0.75	0.35	0.785	0.75	0.75	0.565	0.68	0.5
μ_{ML}	0.75	1	0.65	1	0.1	0.75	0.805	0.795	0.55
μ_{SL}	0.25	0.75	0.25	0.945	0.75	0.75	0.66	0.785	0.35
Φ_M^{μ}	0.375	0.5	0.375	0.37	0.5	0.5	0.35	0.35	0.4
Φ_S^{μ}	0.25	0.25	0.3	0.365	0.375	0.375	0.33	0.33	0.35
Φ_L^{μ}	0.375	0.25	0.325	0.265	0.125	0.125	0.32	0.32	0.25
I_{MS}^{μ}	0	-0.5	0.1	-0.63	-0.75	-0.5	-0.26	-0.26	0.1
I_{ML}^{μ}	0.75	0	0.65	0	0.25	0	0.24	-0.01	0.4
I_{SL}^{μ}	0	0	0	-0.1	0	0.25	-0.01	0.01	0.1

Shapley interaction index $I_{ij}^\mu = \mu_{ij} - \mu_i - \mu_j$

New robust interpretation of interactions

Please, use the “necessary” and “possible” paradigm

- i and j are *necessary complementary* if $\forall \mu, I_{ij}^\mu > 0$.
- i and j are *necessary redundant* if $\forall \mu, I_{ij}^\mu < 0$.
- i and j are *necessary independent (no interaction)* if $\forall \mu, I_{ij}^\mu = 0$.

(Mayag & Bouyssou (2020), Kaldjob et al. (2022))

Preferences and no interactions

Example (A classic example of Grabisch et al. (2010))

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- Let us suppose that the previous preferences change as follows:

$$a \succ_X b \text{ and } c \succ_X d$$

Preferences and no interactions

	1 : Mathematics (M)	2 : Statistics (S)	3 : Language (L)
<i>a</i>	$u_1(16) = 16$	$u_2(13) = 14$	$u_3(7) = 7$
<i>b</i>	$u_1(16) = 16$	$u_2(11) = 11$	$u_3(9) = 9$
<i>c</i>	$u_1(6) = 6$	$u_2(13) = 14$	$u_3(7) = 7$
<i>d</i>	$u_1(6) = 6$	$u_2(11) = 11$	$u_3(9) = 9$

$a \succ_X b$ and $c \succ_X d$ are representable by this additive model F_{add} :

$$F_{add}(a) = u_1(16) + u_2(13) + u_3(7) = 16 + 14 + 7 = 37,$$

$$F_{add}(b) = u_1(16) + u_2(11) + u_3(9) = 16 + 11 + 9 = 36,$$

$$F_{add}(c) = u_1(6) + u_2(13) + u_3(7) = 6 + 14 + 7 = 27,$$

$$F_{add}(d) = u_1(6) + u_2(11) + u_3(9) = 6 + 11 + 9 = 26.$$

$\implies F_{add}$ satisfies the (mutual preference) independence property.

Preferences and no interactions

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$a \succ_X b$ and $c \succ_X d$ (representable by F_{add}) are also representable by a C_μ with non null interactions (even if the independence property is satisfied).

		$\mu(\{1\})$	0.1	I_{12}^μ	0.3
$C_\mu(a)$	12.7	$\mu(\{2\})$	0.5	I_{13}^μ	0.1
$C_\mu(b)$	11.3	$\mu(\{3\})$	0.5	I_{23}^μ	-0.5
$C_\mu(c)$	9.5	$\mu(\{1, 2\})$	0.9	ϕ_1^μ	0.3
$C_\mu(d)$	8.5	$\mu(\{1, 3\})$	0.7	ϕ_2^μ	0.4
		$\mu(\{2, 3\})$	0.5	ϕ_3^μ	0.3

A partial conclusion

- Mutual preference independence **violated** $\implies \exists \mu$ s.t. $I_{ij}^\mu \neq 0$?

(**YES !** ... but not always)

- Mutual preference independence **satisfied** $\implies \forall \mu, I_{ij}^\mu = 0$?

(**NO !**)

Remark

i and j are independent (no interaction) $\iff \forall \mu, I_{ij}^\mu = 0$? (**NO !**)

The Shapley index I_{ij}^μ is not related to the mutual preference independence axiom.

Mutual preference independence property

Definition

S is preference independent of $N \setminus S$, w.r.t. \succsim_X , if for all $x_S, x'_S \in X_S$, $a_{N \setminus S}, b_{N \setminus S} \in X_{N \setminus S}$,

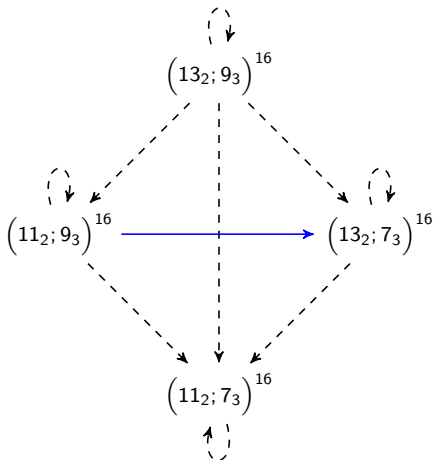
$$(x_S, a_{N \setminus S}) \succsim_X (x'_S, a_{N \setminus S}) \iff (x_S, b_{N \setminus S}) \succsim_X (x'_S, b_{N \setminus S})$$

Using binary relations induced in X_S , we have

$$x_S \succsim_S^{a_{N \setminus S}} x'_S \iff x_S \succsim_S^{b_{N \setminus S}} x'_S$$

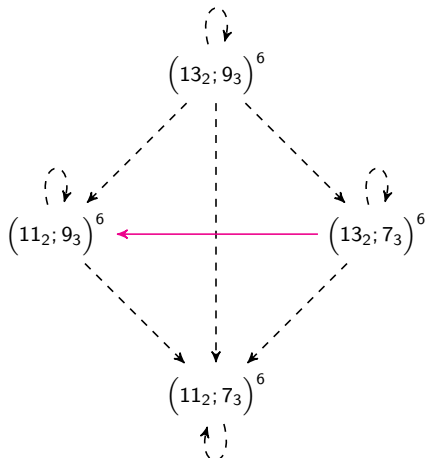
Example (A classic example of Grabisch et al. (2010))

$$(16, 13, 9) \succsim_X (16, 11, 9) \succsim_X (16, 13, 7) \succsim_X (16, 11, 7) \\ \implies (13, 9) \succsim_{23}^{16} (11, 9) \succsim_{23}^{16} (13, 7) \succsim_{23}^{16} (11, 7).$$

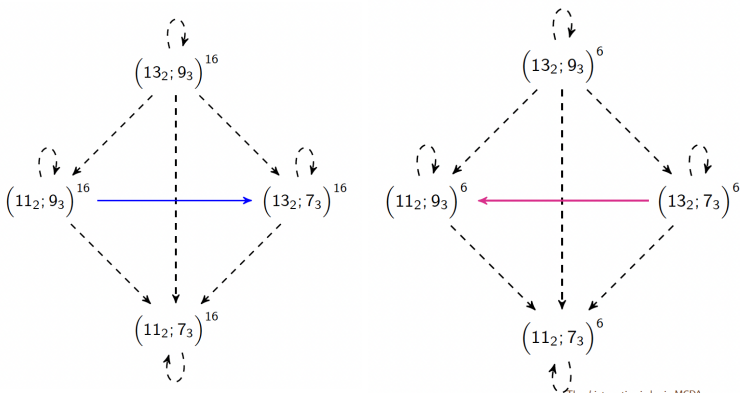


Example (A classic example of Grabisch et al. (2010))

$$(6, 13, 9) \succcurlyeq_X (6, 13, 7) \succcurlyeq_X (6, 11, 9) \succcurlyeq_X (6, 11, 7) \\ \implies (13, 9) \succcurlyeq_{23}^6 (13, 7) \succcurlyeq_{23}^6 (11, 9) \succcurlyeq_{23}^6 (11, 7).$$



Example (A classic example of Grabisch et al. (2010))



$$\text{Kendall distance : } D_K(\succ_{23}^{16}, \succ_{23}^6) = \left| \succ_{23}^{16} \setminus \succ_{23}^6 \right| + \left| \succ_{23}^6 \setminus \succ_{23}^{16} \right| = 2$$

The d -interaction index

$$I^d(S) = \sum_{\{y_{N \setminus S}, z_{N \setminus S}\} \subseteq X_{N \setminus S}} d(\tilde{\lambda}_S^{y_{N \setminus S}}, \tilde{\lambda}_S^{z_{N \setminus S}})$$

- The d -interaction index of a coalition S of criteria captures through the distance the effect of criteria of $N \setminus S$ over S .
- The d -interaction index is unrelated to the preference representation model.
- In the previous example:
 - $I^d(23) = \sum_{\{y,z\} \subseteq X_1} D_K(\tilde{\lambda}_{23}^y, \tilde{\lambda}_{23}^z) = D_K(\tilde{\lambda}_{23}^{16}, \tilde{\lambda}_{23}^6) = 2$
 - $I^d(13) = D_K(\tilde{\lambda}_{13}^{13}, \tilde{\lambda}_{13}^{11}) = 0$ and $I^d(12) = D_K(\tilde{\lambda}_{12}^9, \tilde{\lambda}_{12}^7) = 0$

The d -interaction index

Theorem (Mayag and Tchantcho (2022))

A preference relation \succsim_X satisfies the (mutual preference) independence axiom



$$I^d(S) = 0 \text{ for all } S \in 2^N$$



$$I^d(N \setminus i) = 0 \text{ for all } i \in N.$$

Remark

- $I^d(S) \geq 0$ for all $S \in 2^N$
- *There is no negative interactions related to the independence axiom, in this case.*

The d -interaction index

Example (with binary alternatives: 0-1 values)

HDI index: 1-Life expectancy index (LEI), 2-Educational index (EI) and 3-Adjusted real GDP per capita (GDPI).

$$(1, 1, 0) \sim_X (1, 0, 0) \text{ and } (0, 1, 0) \succ_X (0, 0, 0)$$

We get a necessary negative interaction

$$I_{12}^\mu = \mu(\{1, 2\}) - \mu(\{1\}) - \mu(\{2\}) < 0$$

But

- $(1, 1, 0) \sim_X (1, 0, 0) \implies (1, 0) \sim_{23}^1 (0, 0)$
- $(0, 1, 0) \succ_X (0, 0, 0) \implies (1, 0) \succ_{23}^0 (0, 0)$

$$\text{So } I^d(23) = I^d(\{1, 2, 3\} \setminus \{1\}) = 2 > 0$$

At the end ...

- (mutual preference) independence axiom violated \implies use a non additive model
- (mutual preference) independence axiom violated $\iff \exists i$, s.t., $I^d(N \setminus i) \neq 0$
- **Before talking about interactions (related to the independence axiom), try first to represent the preferences by an additive model**
- **Ethic and transparency aspects:**
 - (We) Provide a clear and robust interpretation of interaction in MCDA
 - Do the negative interactions really exist ?
- The computation of $I^d(S)$ is not simple

At the end ...

Paul Valéry (Artist, Writer, Poet, Philosopher (1871-1945))

Everything simple is false. Everything complex is unusable.

A proverb summarizing the difficulty of a task: if we keep things as simple as possible, we probably forget many special cases; if we try to predict all cases, the result becomes so complex that no one can understand how it works.

Tout ce qui est simple est faux,
mais tout ce qui ne l'est pas est
inutilisable.



Paul Valéry

www.citation-celebre.com