PhD title: "Integer Programming with Bounded Subdeterminants"

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Scope. A large part of optimization problems amounts to optimizing a linear function over a set of integer points satisfying a system of linear inequalities. The structure of the *constraint matrix*, the matrix defining this linear system, has a huge impact on the complexity of the problem. For instance, when this matrix is totally unimodular, then the linear system describes the underlying integer polyhedron. This implies that the problem is easy since it can be solved without considering the integrality requirements.

The past few years, numerous positive results were obtained by bounding the maximum value of the subdeterminants of this matrix [7, 1, 2, 4, 6]. Some of these results show that this bounding might impact the complexity of the associated optimization problem. For instance, it has recently been shown [2] that if the subdeterminants are in $\{0, \pm 1, \pm 2\}$, then the optimization problem is polynomial-time solvable.

The major open question in this topic, which will be the main goal of the thesis, is the following: If the subdeterminants of the constraint matrix are bounded by a constant, can one find an optimal integer solution in polynomial time? A starting point could be the special case of box-totally dual integral polyhedra [5], in which the determinants of the contraint matrix have a very particular structure.

This topic fits perfectly within the research directions of LAMSADE's pole 2, as its mixes combinatorial optimization considerations with matricial and geometrical aspects.

Objectives. For a start, we would like to study the properties of matrices with bounded subdeterminants. Then, intermediate steps composed of specific unsolved subcases will be explored. For instance, we propose the following problems to be studied:

1. Tighten the bound of [3] on the number of columns of such a matrix.
2. Study the case of box-totally dual integral polyhedra.
3. Study the specific cases of 0/1 matrices, for example in packing and covering problems.
4. Study the behaviour of known algorithms in this setting.

These steps will be a good starting point for exploring the general case, which is the aim of this PhD Thesis.

Planning for the first year.

- September-December 2024: Learn the various techniques that involve bounded subdeterminants [1, 2, 4, 6], together with the matricial structure of box-totally dual integral polyhedra [5].
- January-March 2025: Study the columns behavior of matrices with bounded subdeterminants.
- April-November 2025: Apply the recent techniques to box-totally dual integral polyhedra, 0/1 constraint matrices, and packing and covering polytopes.
- End of 2025: Write the paper.

Impact and consequences. Progress towards a polynomial algorithm for integer programming with bounded subdeterminants will provide key insights into the challenges of integer programming. It has the potential to identify parameters that simplify problem-solving and may lead to practical applications. Additionally, the research may extend recent results, solving previously unsolved special cases and contributing to both theoretical foundations and algorithmic efficiency in optimization problems.

Planning for 3 years. The planning for the first year can be found above. For the next two years, the planning will heavily rely on how the results advanced during the first year. Yet, we can imagin the following: at the beginning of 2026, the student will start to investigate the general case. In particular, he will have to study how the structure of the integer points evolves when general matrices come into play, with respect to more classical situations.

Required skills. The Ph.D. student should have a master's degree in optimization or mathematical programming. A solid background in mathematics, especially in linear algebra and polyhedra, will be highly valued. Familiarity with integer programming concepts and techniques is expected, given the optimization focus.

References

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