A outranking-based sorting method for partially ordered categories

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Regrouping actions
Different types of problems

Regrouping :

The problem of regrouping '*similar*' actions has received a lot attention and finds its applications in different fields such as finance, agriculture, marketing, image processing, etc.

Different grouping problems may be encountered.

A possible differentiation of the grouping problems can be done on the basis of the *predefinition* and the *order* on the groups.

Regrouping of actions
Different types of problems

Regrouping actions :

	Not defined a priori	Defined a priori
Not Ordered groups	Clustering *clusters*	Classification *classes*
Ordered groups	Ordered Clustering - Ranking *clusters*	Sorting *categories*

Regrouping of actions
Different types of problems

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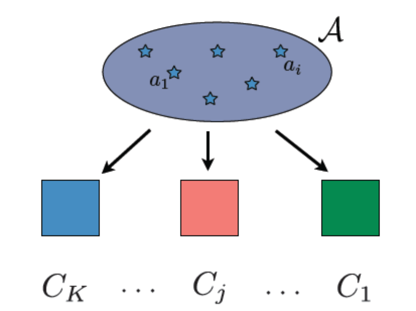
Classification - Sorting
Problem Formalization

Assignment Problems

A set of objects $\mathcal{A} = \{a_1, \ldots, a_n\}$, called actions, have to be assigned to one or several groups of the set $\mathbb{C} = \{C_1, \ldots, C_K\}$, called classes or categories, which are defined a priori by the decision maker (eg. credit demands, movies, etc.).



Classification - Sorting
Problem Formalization

Assignment Problems

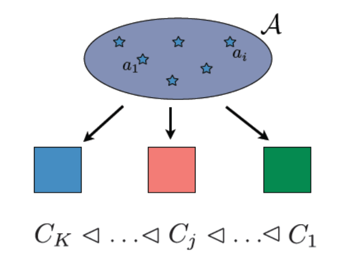
The classes are defined in the way the actions will be treated : actions assigned to the same categories will receive the same treatment (eg. categories of cyclones).

The assignment of an action, does not depend on the assignment of another action : actions are assigned independently (eg. medical diagnosis).

Multicriteria methods
Sorting methods

Completely Ordered categories

The decision maker expresses a complete order among the categories. The categories are defined in a *ordinal* way.
e.g. Credit demand classification - cyclones.



Description of the categories and the actions to be classified by means of a set of criteria $\mathcal{G} = \{g_1, \ldots, g_q\}$.

Several approaches.

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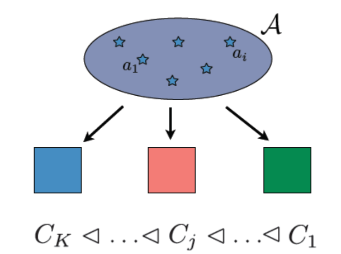


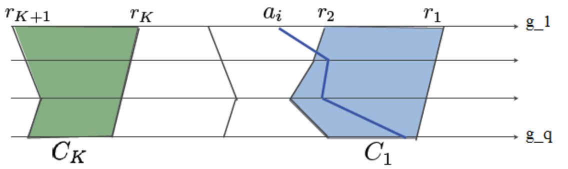
Multicriteria methods
Sorting methods

Completely Ordered categories

Categories are defined by central or limiting profiles, $\mathcal{R} = \{r_1, \ldots, r_{K+1}\}$.

A preference or outranking relation is build between a_i and r_j : $S(a_i, r_j)$ and $S(r_j, a_i)$.

These outranking relations are exploited to assign the action a_i : e.g. Electre-Tri, Filtering Methods, FlowSort, etc.



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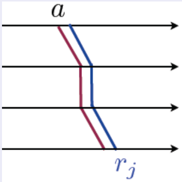

Indifference	$a \mathcal{I} r_j$	$a \equiv r_j$
Incomparbility	$a \mathcal{J} r_j$	$a >< r_j$
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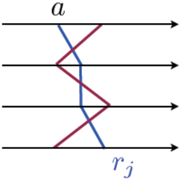

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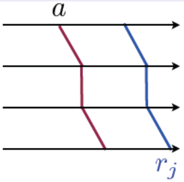

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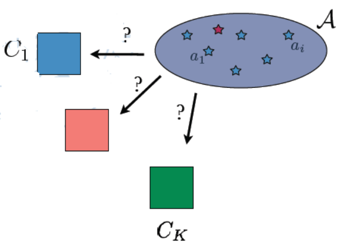
Multicriteria methods
Classification methods

Classes with no preference relation

The decision maker does not express any preference relation amongst the classes.
Classes are defined in a *nominal* way.
e.g. Classification of books.



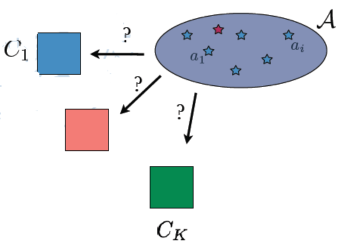
Multicriteria methods
Classification methods

Classes with no preference relation

Description of the classes and the actions to be classified by means of attributes/criteria.

Several approaches.

Use of a similarity index, indifference index computed between the reference profiles and the actions of $\mathcal{A}$.



PROAFTN, TRINOMFC, Filtering methods, etc.

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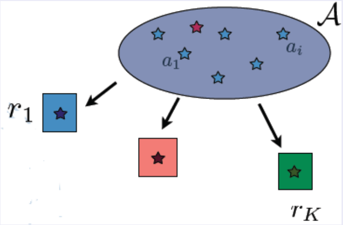

Classes are defined by means of (central) reference profiles $\mathcal{R} = \{r_1, \ldots, r_K\}$.

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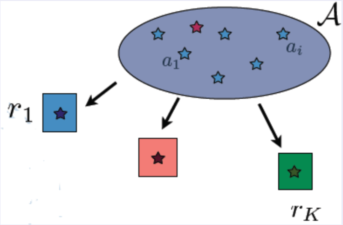

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Multicriteria methods
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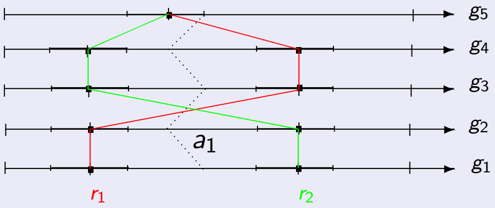
Classes with no preference relation

The central profiles and the actions are defined by a set of attributes/criteria $\mathcal{G} = \{g_1, \ldots, g_q\}$.



For each criterion/attribute $g_j \in \mathcal{G}$ a uni-criterion similarity or indifference index $c_k^l(a_i, r_j)$ is computed.

All these uni-criterion indexes are then aggregated to $\mathcal{I}(a_i, r_j)$.

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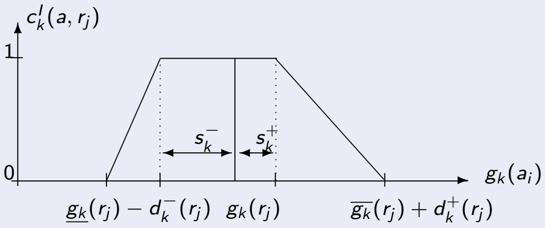

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$$\mathcal{I}(a_i, r_j) = \sum_{k=1}^{q} w_k \times c_k^I(a_i, r_j)$$

$$\forall r_j \in \mathcal{R}$$

Assignment Rules.

Multicriteria methods
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Assignment Rules.

$$a_i \in C_k \Leftrightarrow \begin{cases} I(a_i, r_k) = \max_{\forall r_j \in \mathcal{R}} I(a_i, r_j) \\ I(a_i, r_k) \geq \lambda_I, \ \forall r_j \in \mathcal{R} \end{cases}$$

Aim of this work

- Can we use indifference/similarity based classification methods, which use criteria, when there is a (partial or complete) order on the categories (and vice-versa) ?
- Is there a relationship between indifference-based classification approach and outranking-based sorting approach ? (advantages ?)
- How can we tackle problems when the categories are partially ordered ?

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2 Properties of classification methods

- Preference-orientation dependency property
- Use of indifference-based classification methods in 'sorting' problems

Preference-orientation dependency
Usefulness of 'criteria' in classification problems ?

Some indifference or similarity based classification methods use 'criteria' to define the profiles of non-ordered classes :
e.g. PROAFTN, TRINOMFC, Filtering Methods.

Nevertheless, the indifference or similarity index is (often) symmetric $\Rightarrow$ there might be a loss of information.

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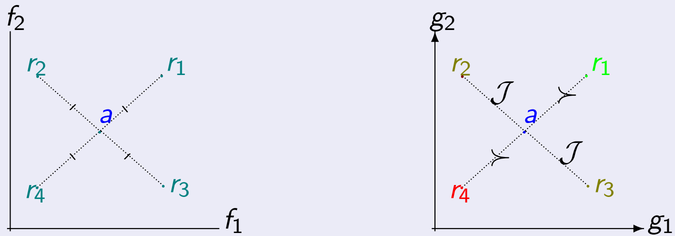

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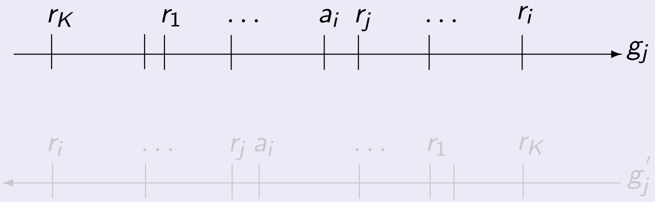
Nevertheless, the indifference or similarity index is (often) symmetric ⇒ there might be a loss of information.

Analyze the use of preference-orientation : does the assignment of an action depend on it ?

Preference-orientation dependency

Inverse Function of a criterion $\mathfrak{I}nv(\mathcal{G}, g_j)$

Let us define the inverse function $\mathfrak{I}nv(\mathcal{G}, g_j)$ associated to a set of criteria $\mathcal{G}$ and criterion g_j, which transforms g_j to $g_j^{'}$ such that the induced preference order on the criterion is inverted.



$$aS_{g_j}b \Leftrightarrow bS_{g_j^{'}}a$$

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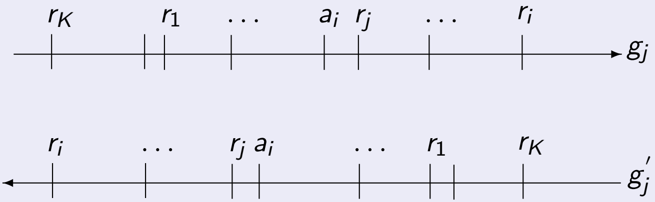

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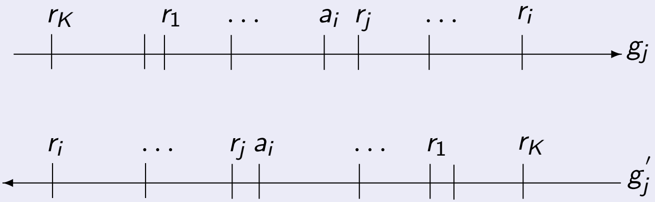

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Preference-orientation dependency

Inverse Function of a criterion $\Im nv(\mathcal{G}, g_j)$

In case of a real-valued criterion, we may define this $\Im nv(\mathcal{G}, g_j)$ function by changing the sign of the criterion values :
$\forall \mathcal{G}, \exists g_j \in \mathcal{G} : \Im nv(\mathcal{G}, g_j) = (\mathcal{G} \setminus \{g_j\}) \cup \{g_j^{'}\}$ where
$g_j^{'}(a_i) = -g_j(a_i), \forall a_i \in \mathcal{A}$

Preference-orientation dependency

Inverse Function of a criterion $\mathfrak{I}nv(\mathcal{G}, g_j)$

Property of preference-orientation dependency

An assignment procedure is preference-orientation dependent, if there exists a set $\mathcal{A}$, such that, inverting one criterion, leads to at least one change in the assignments :

$$\exists \mathcal{A} \mid \exists a_i \in \mathcal{A}, g_j \in \mathcal{G} : C_{\mathcal{S}_\mathcal{G}}(a_i) \neq C_{\mathcal{S}_{\mathcal{G}'}}(a_i) \quad \text{where } \mathcal{G}' = \mathfrak{I}nv(\mathcal{G}, g_j)$$

eg. TRINOMFC, PROAFTN do not respect this property :

$$I(a_i, r_j) = min[S(a_i, r_j), S(r_j, a_i)]$$

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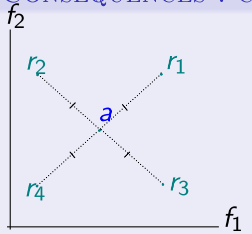
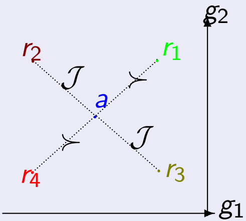
Preference-orientation dependency

Consequences : cases of non-indifference





Different problems :

- categories are defined a priori as dis-similar (nominal classification)
- categories are defined a priori as incomparable (completely non-ordered)

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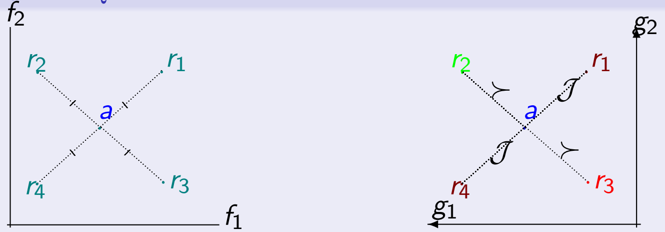

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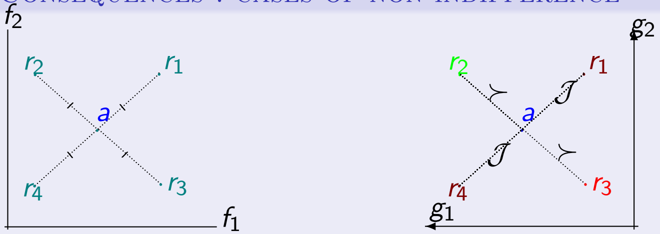

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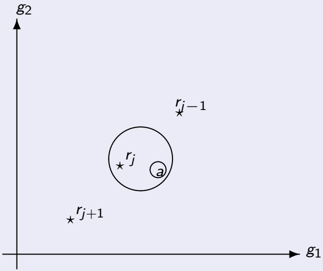
Use of indifference-based classification methods in 'sorting' problems

Consequence-2

Suppose that the categories are completely ordered and defined each by one central profile.

If action *a* is *indifferent* to one or more central profiles, *a* will be assigned to the corresponding category.



Use of indifference-based classification methods in 'sorting' problems

Consequence-2

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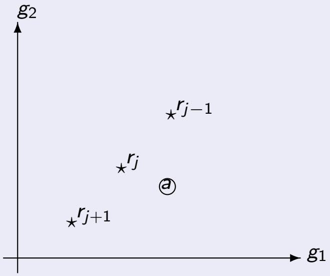
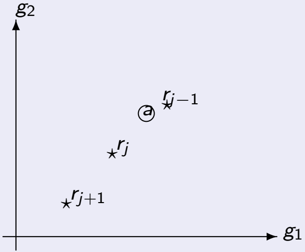
In case of *non-indifference* (incomparability or preference), a will be assigned to none category or to all the categories.





Use of indifference-based classification methods in 'sorting' problems

Consequence-2

Motivation to slightly modify Electre-Tri when working with central profiles while keeping the notion of indifference and distinguishing the cases of non-indifference.

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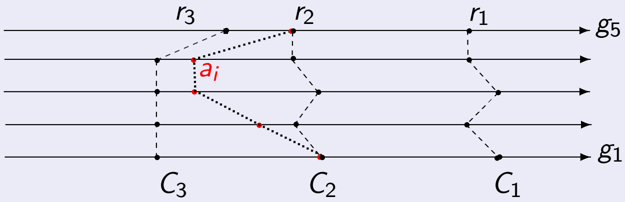
Electre-Tri with central profiles

Context

- Completely Ordered categories, each defined by one central profile r_j
- The central profile respect the order of the categories :
  - $\forall i < j : r_i \succ^D r_j$
  - $\forall i < j : r_i \succ^P r_j$
- Pairwise comparisons between the central profiles and the actions to be sorted are performed by means of outranking relations.



Electre-Tri with central profiles

Assignment rules when working with central profiles

Similar assignment rules are used as when working with limiting profiles, except only the outranking relation S is used for both the optimistic (and not the preference relation) and pessimistic version :

Electre-Tri with central profiles

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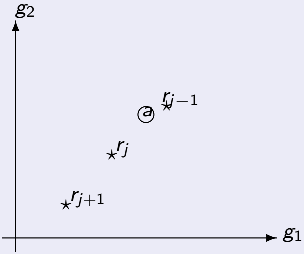
- $r_1 \succ a, r_2 \succ a, \ldots, r_j \succ a, a \succ r_{j+1}, a \succ r_{j+2}, \ldots, a \succ r_K$ *(I)*



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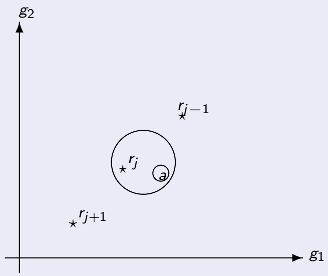

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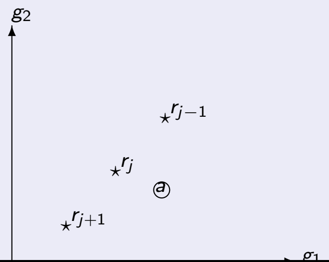

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Electre-Tri with central profiles
Graphical representation

Assignment rules when working with central profiles

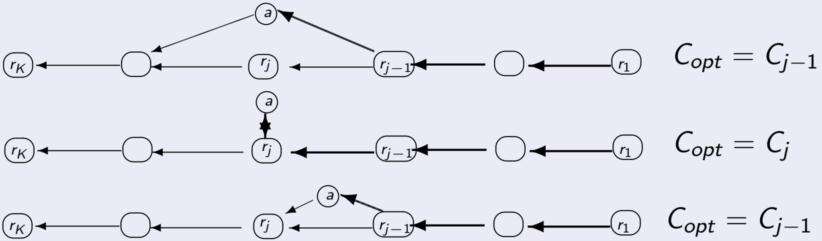

Figure: The reduced "optimistic S-graph" : $xSy \Leftrightarrow x \rightarrow y$

Electre-Tri with central profiles
Graphical representation

Assignment rules when working with central profiles

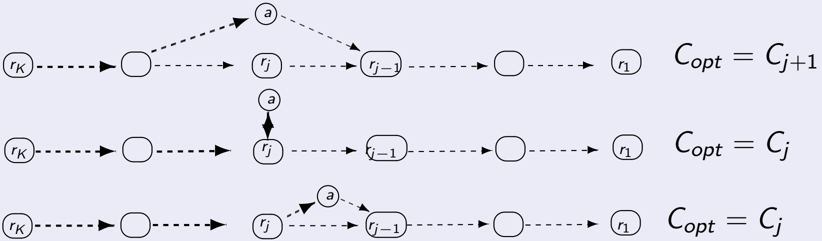

Figure: The reduced "pessimistic S-graph" : $xSy \Leftrightarrow x \leftarrow\!-\,- \; y$

Relation between PROAFTN and Electre-Tri
When working with central profiles

Hypothesis :

Define similarity, discrimination, indifference and preference parameters such that :

$$C^I(a, r_h) = min(S(a, r_h), S(r_h, a)) \quad (1)$$

Relation between PROAFTN and Electre-Tri
When working with central profiles

Proposition :

If the parameters of Electre-Tri-Central and PROAFTN are such that Eq.1 is verified, we have $\forall h \neq 1$ and $h \neq K$:

PROAFTN assigns action a to the unique category C_h

$$\Updownarrow$$

Electre-Tri-Central optimistic and pessimistic affects the action a to the same category C_h.

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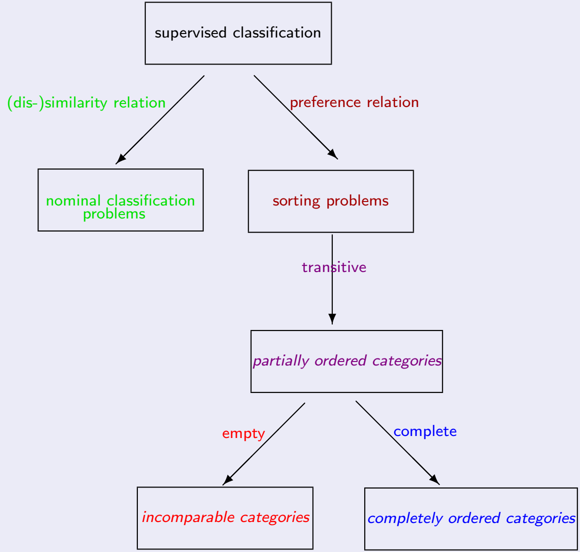
Representation of the different classification problems on the basis of the existing relation between the groups.



Partially Ordered Categories
Example

Recruitment process

The HR wants to evaluate the employees of their computer company according to some profiles and identify four type of persons :

- managers
- engineers
- technical salespeople
- bad-performing employees

Partially Ordered Categories
Example

Recruitment process

The HR wants to evaluate the employees of their computer company according to some profiles and identify four type of persons :

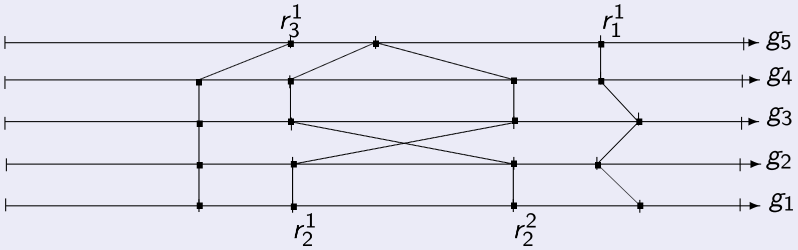
Use of the following criteria :

1. g_1 : software knowledge
2. g_2 : programming experience
3. g_3 : commercial aptitude
4. g_4 : potential mobility
5. g_5 : leadership attitude

Partially Ordered Categories
Example

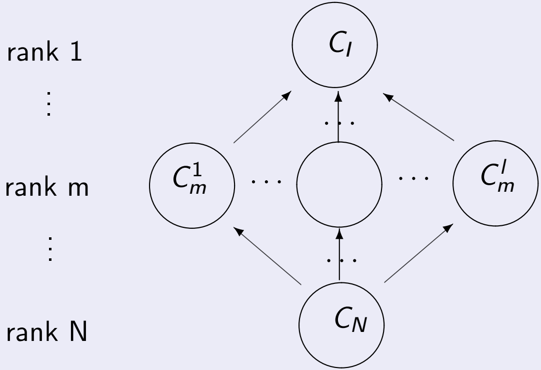
Recruitment process

The HR wants to evaluate the employees of their computer company according to some profiles and identify four type of persons :



Partially Ordered Categories
Example

Recruitment process



Partially Ordered Categories
Assignment Rules

The employees to be categorized, will be pairwise compared to the central reference profiles by means of outranking relations $(S(a_i, r_j^k), S(r_j^k, a_i))$.

The reduced optimistic and pessimistic outranking graphs will be computed.

The assignment of action a_i will depend on its position in these graphs.

Partially Ordered Categories
Assignment Rules

Example

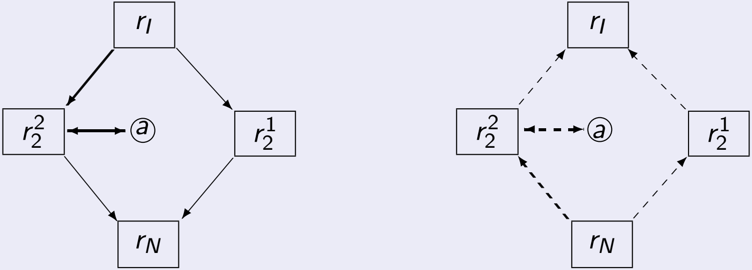


Figure: $C_{opt}(a) = C_2^1$ and $C_{pess}(a) = C_3^1$

Partially Ordered Categories
Assignment Rules

Example

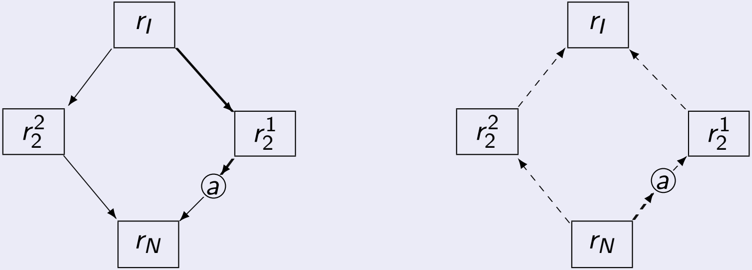


Figure: $C_{opt}(a) = C_2^1$ and $C_{pess}(a) = C_3^1$

Partially Ordered Categories

Particular Subproblems : completely ordered categories

Proposition :

When the categories are completely ordered, the assignment results are the same as the one obtained with Electre-Tri for central profiles.

Partially Ordered Categories

Particular Subproblems : incomparable categories

Proposition :

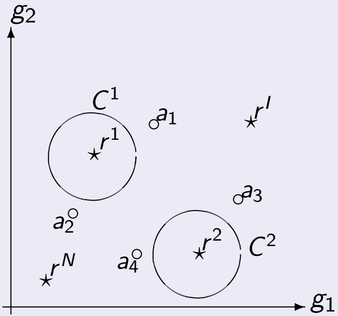
When the categories are incomparable, the assignment results are not always the same as the one obtained with PROAFTN.
The difference lies in the way of how non-indifference is treated.
However, it permits a refinement of the assignments.

Partially Ordered Categories

Particular Subproblems : incomparable categories

Proposition :

When the categories are incomparable, the assignment results are not always the same as the one obtained with PROAFTN.
The difference lies in the way of how non-indifference is treated.
However, it permits a refinement of the assignments.



Conclusions

Conclusions

- Preference-orientation dependency property
- The use of indifference/similarity based classification methods in particular assignment problems
- Electre-Tri with central profiles
- Assignment Rules for a more general assignment problem where the categories are partially ordered.