A state-independent preference representation in he continuous case

David Ríos, Enrique Miranda

Rey Juan Carlos University, University of Oviedo

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The setting

Take a set of alternatives A, a set of states S and a set of consequences C. We consider an order $\succeq$ between the alternatives, so:

- $a \succeq b$ means 'alternative a is preferred to alternative b'.
- $a \succ b$ means 'alternative a is strictly preferred to alternative b'.
- $a \sim b$ means 'alternative a is indifferent to alternative b'.

The idea of an axiomatisation is to provide necessary and sufficient conditions on $\succeq$ to be able to represent it by means of an *expected utility model*.

Some axiomatisations

- L. Savage, *The foundations of statistics*. Wiley, 1954.
- F. Anscombe and R. Aumann, *A definition of subjective probability*. Annals of Mathematical Statistics, 34, 199-205, 1963.
- M. de Groot, *Optimal Statistical Decisions*. McGraw Hill, 1970.

The completeness axiom

The axiomatisations above all require that $\succeq$ is weak order, i.e., complete and transitive: this means in particular that we can express our preferences between any pair of alternatives.

Then we obtain a *unique* utility function u over C and a unique probability p over s such that

$$a \succeq b \Leftrightarrow \int_S \int_C u(c(a,s))p(s)dcds$$

$$\geq \int_S \int_C u(c(b,s))p(s)dcds.$$

Dealing with incomplete information

If we do not have enough information, it is more reasonable that the order between the alternatives is only a quasi-order (reflexive and transitive): there will be alternatives for which we cannot express a preference with guarantees.

$\hookrightarrow$ But then there will not be a unique probability and/or utility representing our information!

Generalisations to imprecise utilities

We consider a unique probability distribution over S and a set U of utility functions over C.

- R. Aumann, *Utility theory without the completeness axiom*. Econometrica 30, 445-462, 1962.
- J. Dubra, F. Maccheroni, E. Ok, *Expected utility theory without the completeness axiom*. Journal of Economic Theory, 115, 118-133, 2004.

Generalisations to imprecise beliefs

We consider a convex set P of probability distributions over S and a unique utility function u.

- D. Ríos Insua, F. Ruggeri, *Robust Bayesian Analysis*. Lecture Notes in Statistics 152. Springer, 2000.
- P. Walley, *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, 1991.
- R. Rigotti, C. Shannon, *Uncertainty and risk in financial markets*. Econometrica, 73, 203–243, 2005.

Imprecise utilities and beliefs

Our goal is to give an axiomatisation for the case where both probabilities and utilities are imprecise, so we have a set P of probabilities and a set U of utilities which are paired up arbitrarily. Some early work in this direction can be found in

- D. Ríos Insua, *Sensitivity analysis in multiobjective decision making*. Springer, 1990.
- D. Ríos Insua, *On the foundations of decision making under partial information*. Theory and Decision, 33, 83-100, 1992.

State dependence and independence

In general the axiomatisations for imprecise beliefs and utilities are made for so-called *state-dependent* utilities, i.e., functions $v : S \times C \to \mathbb{R}$, such that

$$a \succeq b \Leftrightarrow \int_S \int_C v(s, c(a, s)) dcds \geq \int_S \int_C v(s, c(b, s)) dcds \; \forall v \in V.$$

v is called *state-independent* or a *probability-utility pair* when it can be expressed as a product of a probability p over S and a utility U over C:

$$v(s, c) = p(s)u(c) \; \forall s, c.$$

Some state independent representations

- R. Nau, *The shape of incomplete preferences*. Annals of Statistics, 34(5), 2430-2448, 2006.
- T. Seidenfeld, M. Schervisch, J. Kadane, *A representation of partially ordered preferences*. Annals of Statistics, 23(6), 2168-2217, 1995.
- A. García del Amo and D. Ríos Insua, *A note on an open problem in the foundations of statstics*. RACSAM, 96(1), 55-61, 2002.

Nau's framework

- A *finite* set of states S and a *finite* set of consequences C.
- The set $\mathcal{B}$ of horse lotteries $f : S \to \mathcal{P}(C)$.
- H_c denotes the lottery such that $H_c(s)(c) = 1 \; \forall s \in S$.
- 1 denotes the best consequence in C, and 0 the worst.
- For any $E \subseteq S$ and any horse lotteries f, g, $Ef + E^c g$ is the horse lottery equal to $f(s)$ if $s \in E$ and to $g(s)$ is $s \notin E$.

The axioms

(A1) $\succeq$ is transitive and reflexive.

(A2) $f \succeq g \Leftrightarrow \alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h \; \forall \alpha \in (0,1), h$.

(A3) $f_n \succeq g_n \; \forall n, f_n \to f, g_n \to g \Rightarrow f \succeq g$.

(A4) $H_1 \succeq H_c \succeq H_0 \; \forall c$.

(A5) $H_1 \succ H_0$.

A state-dependent representation

$\succeq$ satisfies A1–A5 $\Leftrightarrow$ it is represented by a closed convex set of state-dependent utility functions $\mathcal{V}$, in the sense that

$$f \succeq g \Leftrightarrow U_v(f) \geq U_v(g)\ \forall v \in \mathcal{V},$$

where

$$U_v(f) = \sum_{s \in S, c \in C} f(s,c) v(s,c).$$

A state-independent representation

(A6) If f, g are constant, $f' \succeq g'$, $H_E \succeq H_p$, $H_F \preceq H_q$ with $p > 0$, then

$$\alpha E f + (1-\alpha) f' \succeq \alpha E g + (a - \alpha) g'$$
$$\Rightarrow \beta F f + (1-\beta) f' \succeq \beta F g + (1-\beta) g'$$

for $\beta = 1$ if $\alpha = 1$ and for β s.t. $\frac{\beta}{1-\beta} \leq \frac{\alpha}{1-\alpha}\frac{p}{q}$.

$\succeq$ satisfies (A1)–(A6) if and only if it is represented by a set $\mathcal{V}'$ of state-independent utilities,

$$f \succeq g \Leftrightarrow U_v(f) \geq U_v(g) \forall v \in \mathcal{V}',$$

where $U_v(f) = \sum_{s \in S, c \in C} f(s,c) p(s) u(c)$.

Seidenfeld, Schervisch, Kadane

- A *countable* set of consequences C.
- A *finite* set of states S.
- Horse lotteries $f : S \to \mathcal{P}(C)$, and in particular *simple* horse lotteries, i.e., horse lotteries for which $f(s)$ is a simple probability distribution for all s.
- A strict preference relationship $\succ$ over horse lotteries.

The axioms

(A1) $\succ$ is transitive and irreflexive.

(A2) For any f, g, h, and any $\alpha \in (0,1)$,
$\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h \Leftrightarrow f \succ g$.

(A3) Let $(f_n)_n \to f, (g_n)_n \to g$. Then:

- $f_n \succ g_n \; \forall n$ and $g \succ h \Rightarrow f \succ h$.
- $f_n \succ g_n \; \forall n$ and $h \succ f \Rightarrow h \succ g$.

If $\succ$ satisfies axioms (A1)–(A3), then:

- It can be extended to a weak order $\succeq$ satisfying (A2), (A3).
- $\succ$ is uniquely represented by a (bounded) utility v that agrees with $\succ$ on *simple* horse lotteries.

The representation theorem above is made in terms of state-dependent utilities: any v has associated a probability p and utility functions $u_1, \ldots, u_n$, so that for every horse lottery f,

$$v(f) = \sum_{j=1}^{n} p(s_j) u_j(f(s)).$$

The goal would be to have $u_1 = \ldots, u_n$, i.e., state-independent utilities.

Almost state-independent utilities

$\succ$ admits almost state-independent utilities when for any finite set of rewards $\{r_1, \ldots, r_n\}$, $\epsilon > 0$, there is a pair (p, u_j) s.t. for any $\{s_1, \ldots, s_k\}$ s.t. $\sum_{i=1}^{k} p(s_i) > 1 - \epsilon$,

$$\max_{1 \leq i \leq n, 1 \leq j \neq j' \leq k} |u_j(r_i) - u_{j'}(r_i)| < \epsilon.$$

Some definitions

A state s is $\succ$-*potentially null* when for any horse lotteries f, g with $f(s') = g(s')\ \forall s' \neq s$, $f \sim g$.

We denote f_L the horse lottery which is constant on the probability distribution L over C.

Given a constant horse lottery f_{L_α},

$$f^{\alpha}_{j,m} := \begin{cases} (1 - 2^{-m}) f_0 + 2^{-m} f_{L_\alpha} \text{ if } s \neq s_j \\ f_{L_\alpha} \text{if } s = s_j \end{cases}$$

An (almost) state-independent representation

- (A4) If s_j is not $\succ$ potentially null, then for each acts $f_{L_1}, f_{L_2}, f_1, f_2$, $f_{L_1} \succ f_{L_2} \Leftrightarrow f_1 \succ f_2$, where $f_i(s) = f_i$ if $s = s_j$, $f_1(s) = f_2(s)$ otherwise.
- (A5) For any two constant horse lotteries $f_{L_\alpha}, f_{L_\beta}$, it holds that
$$f_{L_\alpha} \succ f_{L_\beta} \Leftrightarrow f^{\alpha}_{j,m} \succ f^{\beta}_{j,m} \; \forall m \in \mathbb{N}, \forall j.$$

If $\succ$ satisfies (A1)–(A5), then it admits almost state-independent utilites.

Ríos Insua and García del Amo

- A *compact* set $S \subseteq \mathbb{R}^n$ of states.
- A *compact* set $C \subseteq \mathbb{R}^m$ of consequences.
- The set of Young measures $f : S \longrightarrow ca(C)$, where $ca(C)$ are the signed measures of bounded variation on $\mathcal{B}_X$.

The axioms

(A1) $\succeq$ is transitive and reflexive.

(A2) For any f, g, h horse lotteries, $\alpha \in (0, 1)$,
$f \succeq g \Rightarrow \alpha f + (1 - \alpha)h \succeq \alpha g + (1 - \alpha)h$.

(A3) If $f_n \succeq g_n\ \forall n$ and $f_n \to f, g_n \to g$, then $f \succeq g$.

A state-dependent representation

$\succeq$ satisfies (A1)–(A3) if and only if there is a set of state-dependent utilities $\mathcal{V}$ of the form

$$v(s,c) = \sum_{i=1}^{j} u_i(s)p_i(c),$$

with u_i a utility function over S and p_i a density function on C for $i = 1, \ldots, j$, $j \in \mathbb{N}$, such that

$$f \succeq g \Leftrightarrow \int_S \int_C v(s,c)df_s(c)ds \geq \int_S \int_C v(s,c)dg_s(c)dc \forall v$$

The problem

The goal would be to give an axiomatisation of state-independent representations in the context of Ríos Insua and García del Amo, i.e.:

- For a compact set of states S.
- For a compact set of consequences C.

An idea would be to use functional analysis results so that in the above representation we have $j = 1$.

Another idea would be to extend Nau's or Seidenfeld et al.'s results using limit arguments.

Discretising the spaces

For any natural number n, we can consider $\mathcal{S}^n$, $\mathcal{C}^n$ discretisations of $\mathcal{S}, \mathcal{C}$ with diameters smaller than $\frac{1}{2^n}$.

We may also assume without loss of generality that given $n > n'$, $\mathcal{S}^n$ is a refinement of the partition $\mathcal{S}^{n'}$ and $\mathcal{C}^n$ is a refinement of $\mathcal{C}^{n'}$.

We shall denote k_n the number of different elements in the partition $\mathcal{S}^n$ and j_n the total number of elements in the partition $\mathcal{C}^n$.

Relating the horse lotteries (I)

For each natural number n and each set S_n^i in the partition $\mathcal{S}^n$, we select an element s_n^i in S_n^i.

This means just taking a selection U_n of

$$\begin{aligned}\Gamma_n : \mathcal{S} &\rightarrow \mathcal{P}(\mathcal{S}) \\ s &\hookrightarrow S_n^i \Leftrightarrow s \in S_n^i.\end{aligned}$$

We assume that given $n > n'$, the selections $U_n, U_{n'}$ are *consistent*:

$$U_{n'}(s) \in \Gamma_n(s) \Rightarrow U_n(s) = U_{n'}(s).$$

Relating the horse lotteries (II)

Let $\mathcal{F}_n := \mathcal{F}_{\mathcal{S}^n, \mathcal{C}^n}$ denote the set of horse lotteries between $\mathcal{S}^n$ and $\mathcal{C}^n$.

Consider the mapping $\pi_n : \mathcal{F} \to \mathcal{F}_n$ given by

$$\pi_n(f)(S_n^i)(C_n^j) := f(s_n^i)(C_n^j)\ \forall C_n^j \in \mathcal{C}_n, S_n^i \in \mathcal{S}_n.$$

π_n is onto.

Discretising the relationship

Let $\preceq$ be a preference relation on $\mathcal{F}$. Then for each natural number we define a preference relation $\preceq_n$ on $\mathcal{F}_n$ by

$$f \preceq_n g \Leftrightarrow \forall f' \in \pi_n^{-1}(f), g' \in \pi_n^{-1}(g), f \preceq g.$$

1. If $\preceq$ is transitive, so is $\preceq_n$.
2. If $\preceq$ is antisymmetric, so is $\preceq_n$.

But...

1. $\preceq_n$ may not be reflexive, even if $\preceq$ is!
2. $\preceq_n$ may not be a total order, even if $\preceq$ is!

As a consequence,

$$\exists n_0 \in \mathbb{N} \text{ s.t. } \pi_n(f) \preceq_n \pi_n(g) \; \forall n \geq n_0 \Rightarrow f \preceq g$$

but the converse is not necessarily true.

Projecting probabilities and utilities

For any natural number n, let

$$\begin{aligned} H_n : \ \mathcal{U} &\to \ \mathcal{U}_n \\ u &\hookrightarrow \ H_n(u) : \mathcal{C} \to \mathbb{R} \\ & \qquad\qquad\qquad c \hookrightarrow u(c_n^j) \Leftrightarrow c \in C_n^j. \end{aligned}$$

We consider also the functional $T_n : \mathcal{P}_{\mathcal{S}} \to \mathcal{P}_{\mathcal{S}_n}$, given by $T_n(P)(S_n^i) = P(S_n^i)$ for all $S_n^i \in \mathcal{S}_n$.

Properties of H_n, T_n

- For any natural number n, H_n, T_n are onto.

- If we consider on $\mathcal{U}_{\mathcal{C}}$ the topology of uniform convergence and on $\mathcal{U}_n$ the topology of point-wise convergence, then T_n is a continuous mapping for all n.

- If we consider on $\mathcal{P}_{\mathcal{S}}$ the weak-* topology and on $\mathcal{P}_{\mathcal{S}_n}$ the topology of weak convergence, then H_n is a continuous mapping for all n.

If for $\preceq_n$ satisfies the axioms (A1)-(A6) of Nau, there is some set $B_n \times C_n$ of probability/utility pairs (P_n, U_n), where $P_n \in \mathcal{P}_{\mathcal{S}^n}$, $U_n \in \mathcal{U}_{\mathcal{C}^n}$ such that

$$f \preceq_n g \Leftrightarrow E_{P_n,U_n}(f) \leq E_{P_n,U_n}(g) \, \forall (P_n, U_n) \in B_n \times C_n.$$

The idea is to use these to obtain a representation of $\preceq$.

Step by step projection

Let us define the mapping $\pi_{n,n+1} : \mathcal{F}_n \longrightarrow \mathcal{P}(\mathcal{F}_{n+1})$, that assigns to any $f \in \mathcal{F}_n$ the set of horse lotteries in $\mathcal{F}_{n+1}$ satisfying that for any $g \in \pi^{-1}_{n+1}(f')$, $\pi_n(g) = f$.

Let f, g be horse lotteries in $\mathcal{F}_n$, and consider arbitrary $f' \in \pi_{n,n+1}(f), g' \in \pi_{n,n+1}(g)$.

1. $f \preceq_n g \Rightarrow f' \preceq_{n+1} g'$.
2. $f \sim_n g \Rightarrow f' \sim_{n+1} g'$.

We can relate in this way the expected utilities. Let P be a probability measure on $\mathcal{S}$ and u a utility function on $\mathcal{C}$. For any $f \in \mathcal{F}_n$ there is $f' \in \mathcal{F}_{n+1}$ such that

$$E_{(T_n(P),H_n(u))}(f) = E_{(T_{n+1}(P),H_{n+1}(u))}(f').$$

Moreover, $f' \in \pi_{n,n+1}(f)$.

Making the limit

We can prove that $T_n^{-1}(B_n) \subseteq \mathcal{P}_{\mathcal{S}}$ and $H_n^{-1}(C_n) \subseteq \mathcal{U}_{\mathcal{C}}$ are compact for all n.

As a consequence, $\cap_n T_n^{-1}(B_n), \cap_n H_n^{-1}(C_n) \cap \mathcal{U}^*$ are non-empty.

Let $A := \{(P, U) \in \cap_n T_n^{-1}(B_n) \times \cap_n H_n^{-1}(C_n)\}$ be the corresponding set of probability/utility pairs.

Continuous horse lotteries

Let $\mathcal{F}'$ be the set of *continuous* horse lotteries, where we consider the Euclidean distance on $\mathcal{S}$ and the weak-* topology on $\mathcal{P}_{\mathcal{C}}$. This means that for all $f \in \mathcal{F}'$, all $\epsilon > 0$ and all $u \in \mathcal{U}_{\mathcal{C}}$ there is some $\delta > 0$ such that

$$\|s - s'\| < \delta \Rightarrow |E_{f(s)}(u) - E_{f(s')}(u)| < \epsilon,$$

where $E_{f(s)}(u) = \int_{\mathcal{C}} u(c) f(s)(c) dc$.

Representing (a bit) $\preceq$

- For any $(P, U) \in A$ and any horse lottery $f \in \mathcal{F}'$, $E_{(P,U)}(f) = \lim_n E_{(T_n(P), H_n(U))}(\pi_n(f))$.

- For any $f, g \in \mathcal{F}$, $E_{(P,U)}(f) < E_{(P,U)}(g)\ \forall (P, U) \in A \Rightarrow f \preceq g$.

But still there are many problems:

- This approach will only work with horse lotteries satisfying some kind of continuity.
- The definition of $\preceq_n$ is not satisfactory, and as a consequence we do not obtain the converse in the previous theorem.
- There may be problems with finitely versus σ-additive probabilities.

Other approaches

- Trying to work with the *strict* preferences, like Seidenfeld.
- Look for functional analysis results that help generalising the work by Ríos and del Amo.
- ...and any other ideas you may have!