Control and manipulation in weighted voting games

Haris Aziz[1] M. Paterson[1]

[1]Department of Computer Science
&
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Simple Games

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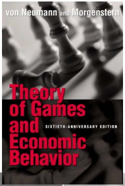
Simple Games

Background: Von Neumann and Morgenstern, *Theory of Games and Economic Behavior, 1944*



John Von Neumann
1903-1957

Oskar Morgenstern
1902-1976.

Simple Games

Reference: A. Taylor and W. Zwicker, *Simple Games: Desirability Relations, Trading, Pseudoweightings*, New Jersey: Princeton University Press, 1999.

> *...few structures arise in more contexts and lend themselves to more diverse interpretations than do simple games.*

Weighted Voting Games

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- Quota, $0 \leq q \leq \sum_{1 \leq i \leq n} w_i$
- A coalition of voters, S is *winning* $\iff \sum_{i \in S} w_i \geq q$
- Notation: $[q; w_1, ..., w_n]$

WVGs are concise although not complete representations of simple games.

Weighted Voting Games - an example

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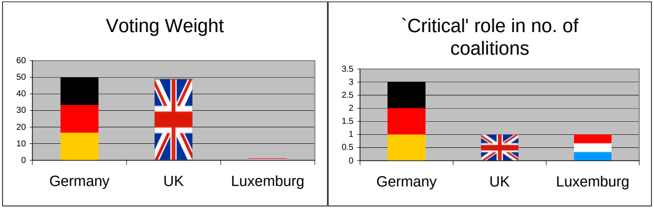
- Winning Coalitions:$\{$Germany, UK, Luxemburg$\}$, $\{$Germany,UK$\}$, $\{$Germany, Luxemburg$\}$
- UK and Luxemburg have the same power!



Weighted Voting Games - Motivation

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Key Concepts

Being critical for a coalition

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Banzhaf Value

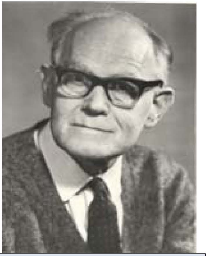
Banzhaf Value, η_i of a player i is the number of coalitions for which i is critical.

Banzhaf Index

Banzhaf Index, β_i is the ratio of the *Banzhaf value* of the player i to sum of the *Banzhaf value* of all players.

Banzhaf Index

John Banzhaf III



Lionel Penrose

Banzhaf Index-Example

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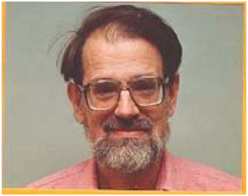
- $\{$Germany, Luxemburg$\}$: critical members are Germany and Luxemburg
- $\{$Germany, Luxemburg, UK$\}$: critical member is Germany.

Banzhaf Index-Example Continued

- Number of coalitions in which Germany is critical: 3
- Number of coalitions in which UK is critical: 1
- Number of coalitions in which Luxemburg is critical: 1
- *Banzhaf index* of Germany is 3/5, *Banzhaf Index* of UK is 1/5 and the *Banzhaf Index* of Luxemburg is 1/5.

Shapley-Shubik index

Depends on permutations instead of coalitions.



Lloyd Shapley

Martin Shubik

Motivation-manipulations in voting

Complexity of Manipulation in voting

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"Would it then be possible to construct a hierarchy reflecting the difficulty of benefiting from strategic behavior?" - Hannu Nurmi, Behavioral Science (1984).

WVGs in recent literature

- Elkind et al., Computing the Nucleolus of weighted voting games. SODA 2009
- M. Zuckerman, P. Faliszewski, Y. Bachrach, and E. Elkind. Manipulating the quota in weighted voting games. AAAI 2008
- E. Elkind, L. Goldberg, P. Goldberg, and M. Wooldridge. On the dimensionality of voting systems. AAAI 2008
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Disadvantageous splitting.

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Unanimity WVGs

Proposition

In a unanimity WVG with $q = w(N)$, if Banzhaf indices are used as imputations of agents in a WVG, then it is beneficial for an agent to split up into agents.

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□

Splitting

Proposition

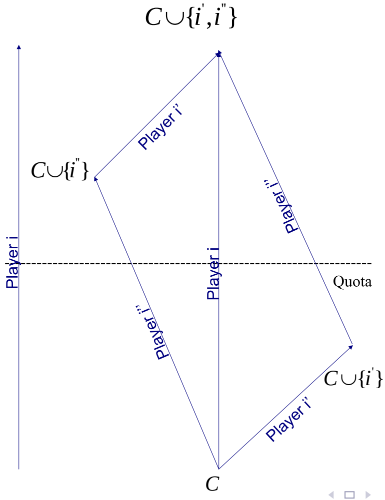
Let WVG v be $[q; w_1, \ldots, w_n]$. If v transforms to v' by the splitting of player i into i' and i'', then $\beta_{i'}(v') + \beta_{i''}(v') \leq 2\beta_i(v)$.

Proof: We assume that a player i splits up into i' and i'' and that $w_{i'} \leq w_{i''}$. We consider a losing coalition C for which i is critical in v. Then $w(C) < q \leq w(C) + w_i = w(C) + w_{i'} + w_{i''}$.

- If $q - w(C) \leq w_{i'}$, then i' and i'' are critical for C in v'.
- If $w_{i'} < q - w(C) \leq w_{i''}$, then i' is critical for $C \cup \{i''\}$ and i'' is critical for C in v'.
- If $q - w(C) > w_{i''}$, then i' is critical for $C \cup \{i''\}$ and i'' is critical for $C \cup \{i'\}$ in v'.

Therefore we have $\eta_{i'}(v') + \eta_{i''}(v') = 2\eta_i(v)$ in each case.

Splitting


C∪{i',i"}
Player i'
C∪{i"}
Player i"
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Quota
Player i"
C∪{i'}
Player i'
C

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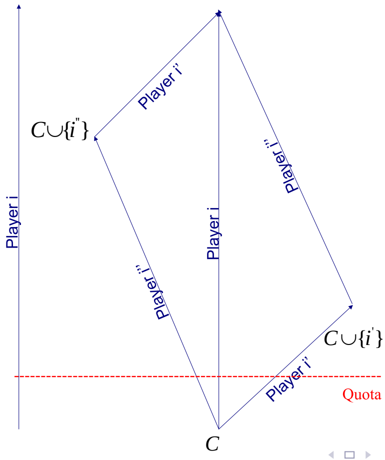

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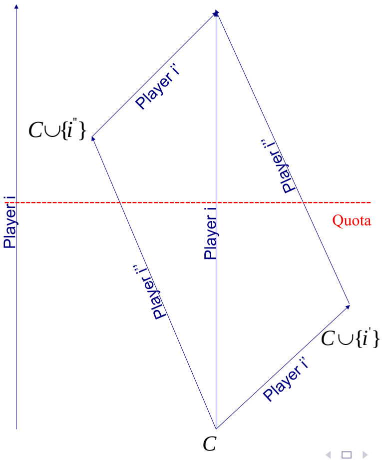

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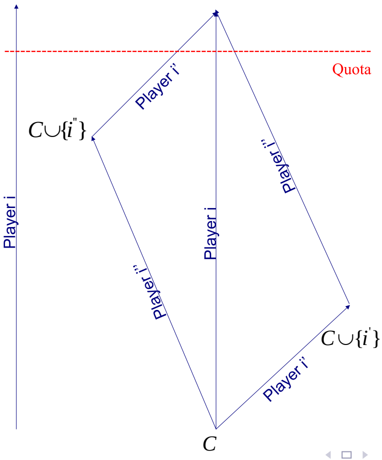

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We have $\eta_{i'}(v') + \eta_{i''}(v') = 2\eta_i(v)$ in each case.

Splitting Proof

Now we consider a player x in v which is other than player i. If x is critical for a coalition C in v then x is also critical for the corresponding coalition C' in v' where we replace $\{i\}$ by $\{i', i''\}$. Hence $\eta_x(v) \leq \eta_x(v')$.

Splitting Proof

Moreover,

$$\begin{aligned}\beta_{i'}(v') + \beta_{i''}(v') &= \frac{2\eta_i(v)}{2\eta_i(v) + \sum_{x \in N(v') \setminus \{i', i''\}} \eta_x(v')} \\ &\leq \frac{2\eta_i(v)}{2\eta_i(v) + \sum_{x \in N(v) \setminus \{i\}} \eta_x(v)} \\ &\leq \frac{2\eta_i(v)}{\eta_i(v) + \sum_{x \in N(v) \setminus \{i\}} \eta_x(v)} = 2\beta_i(v)\end{aligned}$$

Tight bounds

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- Therefore

$$\beta_1 = \frac{n + \binom{n}{2}}{n + \binom{n}{2} + n(1 + \binom{n-1}{2})} = \frac{n+1}{(n-2)^2} \sim 1/n.$$

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- In case player 1 splits up into $1'$ and $1''$ with weights 1 each, then for all players j, $\beta_j = \frac{1}{n+2}$.
- Thus for large n, $\beta_{1'} + \beta_{1''} = \frac{2}{n+2} \sim 2\beta_1$.

Extreme example of disadvantageous split

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- Take WVG v on n players where $v = [3n/2; 2n, 1, \ldots, 1]$ and n is even. Then Player 1 is a dictator.

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- Consider the case where v changes into v' with player 1, splitting up into $1'$ and $1''$ with weight n each.
- For player $1'$ to be critical for a losing coalition in v', the coalition must exclude $1''$ and have from $n/2$ to $n-1$ players with weight 1 or it must include $1''$ and have from 0 to $(n/2)-1$ players with weight 1. So $\eta_{1'}(v') = \eta_{1''}(v') = \sum_{i=0}^{n} \binom{n-1}{i} = 2^{n-1}$.

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- For a smaller player x with weight 1 to be critical for a coalition in v', the coalition must include only one of $1'$ or $1''$ and $(n-2)/2$ of the $n-2$ other smaller players. So, $\eta_x(v') = 2\binom{n-2}{(n-2)/2} \approx \sqrt{\frac{2}{\pi(n-2)}} 2^{n-1}$.

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- Consider the case where v changes into v' with player 1, splitting up into $1'$ and $1''$ with weight n each.
- For player $1'$ to be critical for a losing coalition in v', the coalition must exclude $1''$ and have from $n/2$ to $n-1$ players with weight 1 or it must include $1''$ and have from 0 to $(n/2)-1$ players with weight 1. So $\eta_{1'}(v') = \eta_{1''}(v') = \sum_{i=0}^{n} \binom{n-1}{i} = 2^{n-1}$.
- For a smaller player x with weight 1 to be critical for a coalition in v', the coalition must include only one of $1'$ or $1''$ and $(n-2)/2$ of the $n-2$ other smaller players. So, $\eta_x(v') = 2\binom{n-2}{(n-2)/2} \approx \sqrt{\frac{2}{\pi(n-2)}}2^{n-1}$.

Extreme example of disadvantageous split

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$$\begin{aligned}\beta_{i'}(v') &= \beta_{i''}(v') \\ &\approx \frac{2^{n-1}}{2^{n-1}+2^{n-1}+(n-1)\sqrt{\frac{2}{\pi(n-2)}}2^{n-1}}\end{aligned}$$

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- It is #P-hard for a manipulator to find the ideal splitting to maximize his payoff.
- An easier question is to check whether a beneficial splitting exists or not.
- We define a Banzhaf version of the BENEFICIAL SPLIT problem:
  Name: BENEFICIAL-BANZHAF-SPLIT
  Instance: (v, i) where v is the WVG $v = [q; w_1, \ldots, w_n]$ and player $i \in \{1, \ldots, n\}$
  Question: Is there a way for player i to split his weight w_i between sub-players $i_1, \ldots, i_m$ so in the new game v', $\sum_{j=1}^{k} \beta_{i_k}(v') > \beta_i(v)$?

Complexity of finding a beneficial split

Proposition

BENEFICIAL-BANZHAF-SPLIT is NP-hard even if a player can only split into two players with equal weights.

We prove this by a reduction from an instance of the classical NP-hard PARTITION problem to BENEFICIAL-BANZHAF-SPLIT.
Name: PARTITION
Instance: A set of k weights $A = \{a_1, \ldots, a_k\}$
Question: Is it possible to partition A, into two subsets $P_1 \subseteq A$, $P_1 \subseteq A$ so that $P_1 \cap P_2 = \emptyset$ and $P_1 \cup P_2 = A$ and $\sum_{a_i \in A_1} a_i = \sum_{a_i \in A_2} a_i$.

Proof (NP-hard to decide whether split beneficial)

Reduction

- Given an instance of PARTITION $\{a_1, \ldots, a_k\}$, we can transform it to a WVG $v = [q; w_1, \ldots, w_n]$ with $n = k + 1$ where $w_i = 8a_i$ for $i = 1$ to $n - 1$, $w_n = 2$ and $q = 4\sum_{i=1}^{k} a_i + 2$.
- After that, we want to see whether it can be beneficial for player n with weight 2 to split into two sub-players n and $n + 1$ each with weight 1 to form a new WVG $v' = [q; w_1, \ldots, w_{n-1}, 1, 1]$.

- If A is a 'no' instance of PARTITION, then we see that no subset of the weights $\{w_1, \ldots, w_{n-1}\}$ can sum to $4\sum_i a_i$. This implies that player n is a dummy.
- Even after splitting, new players remain dummies.
- Thus 'no' instance of PARTITION implies a 'no' instance of BENEFICIAL-BANZHAF-SPLIT.

Proof (NP-hard to decide whether split beneficial)

- Now let us assume that A is a 'yes' instance of PARTITION.
- Then after some technical work it can be shown that this implies a 'yes' instance of BENEFICIAL-BANZHAF-SPLIT.

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- We devise a polynomial time algorithm to find a beneficial split if the weights of players are polynomial in n and the player i in question can split into upto a constant k number of sub-players.
- Whenever player i in WVG v splits up according to a split s, we denote the new game by $v_{i,s}$.

How to manipulate?

Algorithm 1 BeneficialSplitInWVG

Input: (v, i) where $v = [q; w_1, \ldots, w_n]$ and i is the player which wants to split into maximum of k sub-players.
Output: Returns NO if there is no beneficial split. Otherwise returns the optimal split $(w_{i_1}, \ldots, w_{i_{k'}})$ where $k' \leq k$, and $\sum_{j=1}^{k'} w_{i_j} = w_i$

BeneficialSplitExists = **false**; BestSplit = $\emptyset$; BestSplitValue = $-\infty$
β_i = BanzhafIndex(v, i)
for $j = 2$ to k **do**
    for Each possible split s where $w_i = w_{i_1} + \ldots + w_{i_j}$ **do**
        SplitValue = $\sum_{a=1}^{j}$ BanzhafIndex$(v_{i,s}, i_a)$
        if SplitValue $>$ β_i **then**
            BeneficialSplitExists = **true**
            if SplitValue $>$ BestSplitValue **then**
                BestSplit = s; BestSplitValue = SplitValue
            end if
        end if
    end for
end for
if BeneficialSplitExists = **false then**
    return **false**
else
    return BestSplit
end if

How to manipulate?

Proposition

Algorithm 1 has computational complexity which is pseudo-polynomial in n

Proof.

It is clear that for a constant k, the number of splits of player i is less than $(w_i)^k$ which is a polynomial in n. Since the computational complexity for each split is also a polynomial in n, therefore Algorithm 1 is polynomial in n if weights are polynomial in n. □

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- (Freixas and Puente): *amplitude* is the maximum μ such that $f_{(\lambda_1,\ldots,\lambda_n),\Lambda}$ is a representation of v whenever $\mathrm{Max}(|\lambda_1|,\ldots,|\lambda_n|,|\Lambda|) < \mu(v)$.

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- For a strict representation of a WVG $[q; w_1,\ldots,w_n]$, for each coalition $S \subseteq N$, let $a(S) = |w(S) - q|$ and $b(S) = q + w(S)$.

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The amplitude is a more precise and accurate indicator of the maximum possible variation than tolerance.

Amplitude

We let WVG-STRICT be the problem of checking whether a WVG $v = [q; w_1, \ldots, w_n]$ is strict or not, i.e., WVG-STRICT $= \{v\colon\ v \text{ is strict}\}$.

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Proposition

WVG-STRICT is co-NP-complete

Proof.

- WVG-NOT-STRICT is in NP since a certificate of weights can be added in linear time to confirm that they sum up to q.

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- Hence WVG-NOT-STRICT is NP-complete and WVG-STRICT is co-NP-complete.

Amplitude

Corollary

The problem of checking whether the amplitude of a WVG is 0 is NP-hard.

Amplitude

Proposition

The problem of computing the amplitude of a WVG v is NP-hard, even for integer WVGs.

Proof.

- Let us assume that weights in v are even integers whereas the quota q is an odd integer $2k-1$ where $k \in \mathbf{N}$.

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The problem of computing the amplitude of a WVG v is NP-hard, even for integer WVGs.

Proof.

- Let us assume that weights in v are even integers whereas the quota q is an odd integer $2k - 1$ where $k \in \mathbf{N}$.
- Then the minimum possible difference between q and A, the weight of the maximal losing coalition, or q and B, the weight of minimal winning coalition is 1.

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- So $A \leq 2k - 2$ and $B \geq 2k$.

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- We see that $\mu(v) \leq 1/2k$ if and only if there exists a coalition C such that $w(C) = 2k$.

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- So $A \leq 2k - 2$ and $B \geq 2k$.
- We see that $\mu(v) \leq 1/2k$ if and only if there exists a coalition C such that $w(C) = 2k$.
- Thus the problem of computing $\mu(v)$ of a WVG is NP-hard by a reduction from the SUBSET-SUM problem.

Amplitude

A similar proof can be used to prove the following proposition:

Proposition

The problem of computing the tolerance of a strict WVG is NP-hard.

Uniform WVGs

Proposition

For a strict representation of a proper uniform WVG $v = [q; \underbrace{w, \ldots, w}_{n}]$, $\tau(v) \leq \frac{1}{3n}$.

Proof:

- Since $\frac{q-A}{q+w(N)} = 1 - \frac{w(N)+A}{q+w(N)}$ is an increasing function of q and $\frac{B-q}{q+w(N)}$ is a decreasing function of q, the tolerance reaches its maximum when $q - A = B - q$, i.e. when q is the arithmetic mean $\frac{A+B}{2}$.
- We let the size of the maximal losing coalition be r and the size of the minimal winning coalition be $r + 1$.

Uniform WVGs

- Then the weight of a maximal losing coalition is rw and the weight of the minimal winning coalition is $(r+1)w$ and $m = w/2$. Since v is proper, $q \geq \frac{1}{2}(nw)$, and $M = q + w(N) \geq \frac{3nw}{2}$
- Then,

$$\tau(v) = m/M \leq \frac{1}{3n}.$$

Uniform WVGs

Proposition

For a uniform WVG $v = [q; \underbrace{w, \ldots, w}_{n}]$, *we have* $B = w\lceil \frac{q}{w} \rceil$ *and* $A = B - w$. *Then,*

$$\mu(v) = \begin{cases} \frac{q-A}{A+q}, & \text{if } q \leq \sqrt{AB} \\ \frac{B-q}{B+q}, & \text{otherwise.} \end{cases}$$

Uniform WVGs

Proof.

- It is clear that B, the weight of the minimal winning coalition is $w\lceil \frac{q}{w} \rceil$ and A, the weight of the maximal losing coalition is $B - w$.

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Proof.

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- For losing coalitions with weight w, $\frac{q-w}{q+w}$ is a decreasing function for w.

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- For losing coalitions with weight w, $\frac{q-w}{q+w}$ is a decreasing function for w.
- For winning coalitions with weight w, $\frac{w-q}{q+w} = 1 - \frac{2q}{q+w}$ is an increasing function for w.

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- Thus if $q \leq \sqrt{AB}$, $\mu(v) = \frac{q-A}{A+q}$.
- Otherwise, $\mu(v) = \frac{B-q}{B+q}$.

□

Unanimity WVGs

Proposition

For a unanimity WVG $v = [q; w_1, \ldots, w_n]$, $\tau(v) \leq \frac{w_n}{4w(N)-w_n} \leq \frac{1}{4n-1}$.

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- We know that $B = w(N)$ and $A = w(N) - w_n$ which means that $w(N) - w_n < q \leq w(N)$.

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- For maximum tolerance, $q = \frac{A+B}{2} = w(N) - \frac{w_n}{2}$.
- Therefore $m = w_n/2$ and $M = w(N) - \frac{w_n}{2} + w(N)$.

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- Therefore $m = w_n/2$ and $M = w(N) - \frac{w_n}{2} + w(N)$.
- Then the tolerance of v satisfies:

$$\tau(v) \leq \frac{m}{M} = \frac{w_n}{4w(N) - w_n} \leq \frac{1}{4n-1},$$

since $w_n \leq w(N)/n$.

Unanimity WVGs

Proposition

For a unanimity WVG $v = [q; w_1, \ldots, w_n]$, $\tau(v) \leq \frac{w_n}{4w(N)-w_n} \leq \frac{1}{4n-1}$.

Proof.

- We know that $B = w(N)$ and $A = w(N) - w_n$ which means that $w(N) - w_n < q \leq w(N)$.
- For maximum tolerance, $q = \frac{A+B}{2} = w(N) - \frac{w_n}{2}$.
- Therefore $m = w_n/2$ and $M = w(N) - \frac{w_n}{2} + w(N)$.
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Thank You

Thank you for your attention. If you are interested in my work, please feel free to email to haris.aziz@warwick.ac.uk

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