Reducing graph coloring to stable set without symmetry

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SPOC 11

Coloring and perfect 0-1 matrices

- $(P) = \{Ax \leq b\}$ is *TDI:*

$$\begin{aligned} & \min\{y^\top b : y^\top A \geq w^\top\} \\ = \; & \min\{y^\top b : y^\top A \geq w^\top, \, y \text{ integer}\} \qquad (\forall w \text{ integer}) \end{aligned}$$

- $P = \{x : Ax \leq b\}$ is *integer:*

$$\begin{aligned} & \max\{w^\top x : x \in P\} \\ = \; & \max\{w^\top x : x \in P, \, x \text{ integer}\} \qquad (\forall w) \end{aligned}$$

- A is 0-1: the clique-matrix $\{\text{maximal cliques}\} \times \{\text{vertices}\}$

$$\begin{aligned} \alpha(G) &:= \max\{\mathbf{1}^\top x : Ax \leq \mathbf{1}, \, x \geq \mathbf{0}, \, x \text{ integer}\} =: \omega(\overline{G}) \\ \overline{\chi}(G) &:= \min\{y^\top \mathbf{1} : y^\top A \geq \mathbf{1}^\top, \, y \geq \mathbf{0}, \, y \text{ integer}\} =: \chi(\overline{G}) \end{aligned}$$

Compact LP formulation

$$
\begin{aligned}
\chi(G) = \min \quad & \textstyle\sum_i x_i \\
\text{s.t.} \quad & x_i \geq x_i^v & (\forall i, \forall v) \\
& \textstyle\sum_i x_i^v = 1 & (\forall v) \\
& x_i^u + x_i^v \leq 1 & (\forall i, \forall uv) \\
& x \geq \mathbf{0} \\
& x \text{ integer}
\end{aligned}
$$

where $G = (V, E)$ is a graph, $i \in \{1, \ldots, |V|\}$, $v \in V$, $uv \in E$.

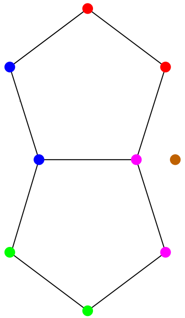
$\hat{G} = G_1 \Box \ldots \Box G_{|V(G)|}$ add $u_1, \ldots, u_{|V(G)|}$ universal

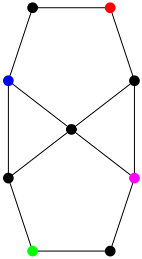
$$\chi(G) + \alpha(\hat{G}) = 2|V(G)|$$

Gallai identities

For all G (without isolated vertex)

$$\rho(G) + \nu(G) = |V(G)| = \alpha(G) + \tau(G)$$





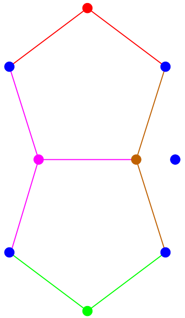
If G has no 3-cycle

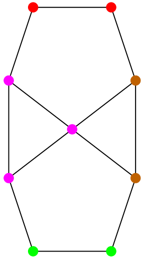
$$\overline{\chi}(G) + \alpha(L(G)) = |V(G)|$$

Gallai identities

For all G (without isolated vertex)

$$\rho(G) + \nu(G) = |V(G)| = \alpha(G) + \tau(G)$$





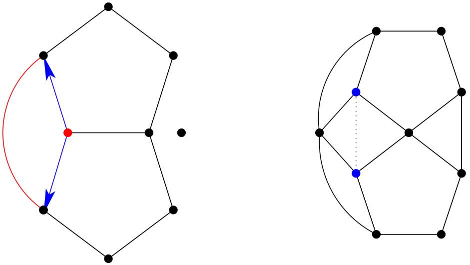
If G has no 3-cycle

$$\overline{\chi}(G) + \alpha(L(G)) = |V(G)| = \alpha(G) + \overline{\chi}(L(G))$$

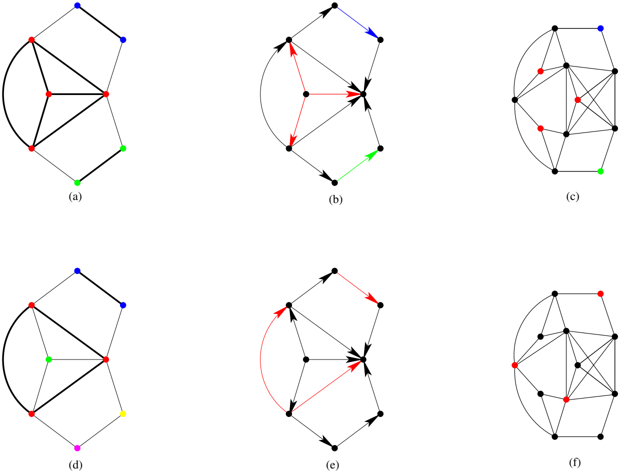
The "sandwich" line-graphs $S(G)$ of G

Choose a orientation $\vec{G}$ of G without 3-dicycle (cliques are acyclic)

$S(\vec{G})$ is the line-graph $L(G)$ of G minus the edges corresponding to simplicial pairs of arcs



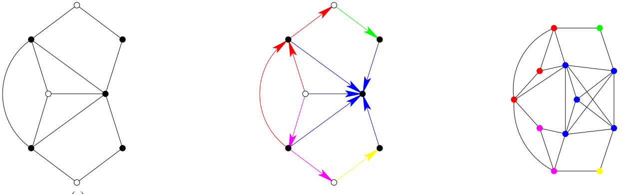
Chromatic Gallai identities



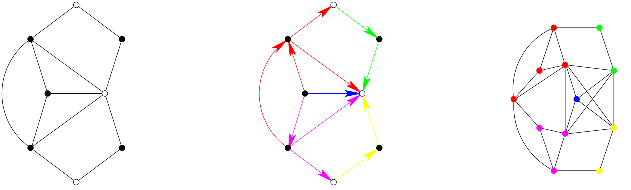
$\vec{G}$ is an orientation of G, without 3-dicycle

$$\overline{\chi}(G) + \alpha(S(\vec{G})) = |V(G)|$$

Chromatic Gallai identities



(a) (b) (c)



(d) (e) (f)

$\vec{G}$ is an orientation of G, without 3-dicycle

$$\overline{\chi}(G) + \alpha(S(\vec{G})) = |V(G)| = \alpha(G) + \overline{\chi}(S(\vec{G}))$$

The operator Φ

Let $\beta(\cdot)$ be a graph parameter such that $\alpha(G) \leq \beta(G) \leq \overline{\chi}(G)$ $(\forall G)$

Choose an orientation $\vec{G}$ of G without 3-dicycle and define

$$\Phi_\beta(G) := |V(G)| - \beta(S(\vec{G}))$$

Corollary. $\alpha(G) \leq \Phi_\beta(G) \leq \overline{\chi}(G)$

Proof.

$$|V(G)| - \overline{\chi}(S(\vec{G})) \leq |V(G)| - \beta(S(\vec{G})) \leq |V(G)| - \alpha(S(\vec{G}))$$

Fractional clique-cover (co-chromatic) number

$$\overline{\chi}_f(G) = \begin{cases} \max & \mathbf{1}^\top x \\ \text{s.t.} & x(K) \leq 1 \quad (\forall \text{clique } K) \\ & x_v \geq 0 \quad (\forall v \in V) \end{cases}$$

$$= \begin{cases} \min & \mathbf{1}^\top y \\ \text{s.t.} & \sum_{K \ni v} y_K = 1 \quad (\forall v) \\ & y_K \geq 0 \quad (\forall \text{clique } K) \end{cases}$$

Lemma. y optimal for $G \Rightarrow x$ feasible for $S(\vec{G})$

$$x_{uv} := \sum_{\text{simplicial stars } S_K \ni uv} y_K$$

Lovász Theta function

$$\vartheta(G) = \begin{cases} \max & \sum_v \|x_v\|^2 \\ \text{s.t.} & \|x_o\|^2 = 1 \\ & x_o^\top x_v = \|x_v\|^2 \qquad (\forall v) \\ & x_u^\top x_v = 0 \qquad (\forall uv \in E) \end{cases}$$

$$= \begin{cases} \min & \|y_o\|^2 \\ \text{s.t.} & \|y_v\|^2 = 1 \qquad (\forall v) \\ & y_o^\top y_v = 1 \qquad (\forall v) \\ & y_u^\top y_v = 0 \qquad (\forall uv \notin E) \end{cases}$$

Lemma. $x \in \mathbb{R}^{d \times m+1}$ optimal for $S(\vec{G}) \Rightarrow y \in \mathbb{R}^{nd \times n+1}$ feasible for G

$$y_{o\,v} := x_o - \sum_{u:uv \in E(\vec{G})} x_{uv} \text{ and } y_{v\,u} := \begin{cases} y_{o\,v} & \text{if } u = v \\ x_{uv} & \text{if } uv \in E(\vec{G}) \\ 0 & \text{otherwise} \end{cases}$$

Improving Lovász's ϑ bound for coloring

The sandwich theorem (Lovász 1979)

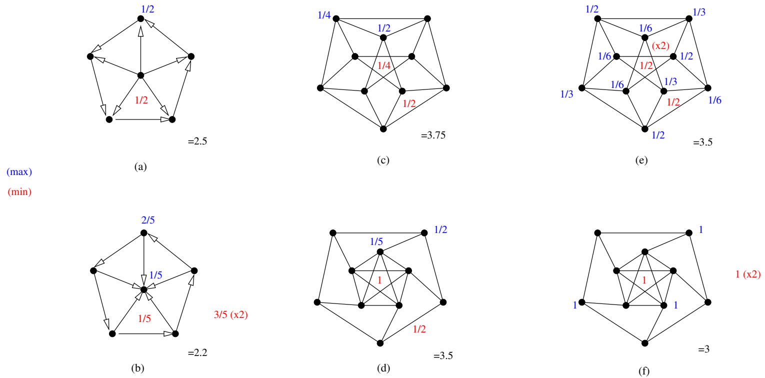
$$\alpha(G) \leq \vartheta(G) \leq \overline{\chi}_f(G) \leq \overline{\chi}(G) \qquad (\forall G)$$

Theorem (C. and Meurdesoif 2014)

$$\alpha(G) \leq \Phi_{\overline{\chi}_f}(G) \leq \overline{\chi}_f(G) \quad \text{and} \quad \vartheta(G) \leq \Phi_{\vartheta}(G) \leq \overline{\chi}(G) \qquad (\forall G)$$

	$\lvert V\rvert$	$\lvert E\rvert$	min $\rho := \frac{\Phi_\vartheta - \vartheta}{\vartheta}$	mean ρ	max ρ
$\overline{M_3}$	5	5	–	23.6%	–
$\overline{M_4}$	11	35	26.8%	27.3%	28.7%
$\overline{M_5}$	23	182	26.5%	27.6%	29.5%
$\overline{M_6}$	47	845	26.0%	27.8%	29.5%

Impact of orientation



	2	2.2	2.25	2.5	3
6-wheel	α	$\overline{\chi}'_f$		$\overline{\chi}_f$	$\overline{\chi}$
orientation (a)	$\Phi_{\overline{\chi}}$		$\Phi_{\overline{\chi}_f}$	$\Phi_{\overline{\chi}'_f}$	Φ_{α}
orientation (b)	$\Phi_{\overline{\chi}}$			$\Phi_{\overline{\chi}_f}$	$\Phi_{\overline{\chi}'_f} = \Phi_{\alpha}$

Complexity

Theorem. (Grötschel, Lovász and Schrijver 1981)

$\vartheta(G)$ can be approximated in polynomial time (for any $\varepsilon > 0$)

Theorem. (Gvozdenović and Laurent 2008)

Computing $\overline{\chi}_f(G)$ is NP-hard

In fact, [GL08] proved that,
given any parameter $\beta(G)$ such that $\beta \in [\overline{\chi}_f, \overline{\chi}]$,
then computing $\beta(G)$ is NP-hard.

Finding α with $\Phi_{\overline{\chi}_f}$ (another proof for GL08)

$\vec{G}_k = G$ orient and add $s_1, \ldots, s_k$ universal sources

Feasible solution x for $\overline{\chi}_f(S(\vec{G}_k))$ (max)

$$x_{uv} := \begin{cases} \frac{1}{\alpha(G)} & \text{if } u = s_i \\ 0 & \text{otherwise} \end{cases}$$

Let $\beta \in [\overline{\chi}_f, \overline{\chi}]$. Start with $k := 1$ and grow it until $\Phi_\beta(\vec{G}_k) = k$.

$\Phi_\beta(\vec{G}_k) = k \iff \alpha(G) = k$

Proof.

$$k \leq \alpha(G) = \alpha(G_k) \leq \Phi_\beta(\vec{G}_k) \leq \Phi_{\overline{\chi}_f}(\vec{G}_k) \leq k + |V(G)| - k\frac{|V(G)|}{\alpha(G)}$$