Short Course on Submodular Functions
Part 2: Extensions and Related Problems
Session 2.A: Partitions

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Partitions

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- $\emptyset \neq P_i \subseteq E$ for all i,
- $P_i \cap P_j = \emptyset$ for all $i \neq j$, and
- $\cup_{i=1}^{n} P_i = E$

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Optimum Partition Problems:

- given E and f
- find a partition $\mathbf{P}$ with minimum cost $f(\mathbf{P})$
  (subject to possible restrictions on the number $k = |\mathbf{P}|$ of parts)

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Clustering

- E is a set of items to be classified
- $f(S)$ is the (negative of) the value of *cluster* S, reflecting
  - the similarities within S, and
  - the dissimilarities with $N \setminus S$

Applications (2)

Multi-layer VLSI Circuit Design (Netlist Partitioning)

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VLSI Circuit Design

Given hypergraph (E, H) with edge weights w_h ($h \in H$), the *hypergraph cut function*

$$f(S) = \sum \{w_h \ : \ h \cap S \neq \emptyset \ \text{ and } \ h \setminus S \neq \emptyset\}$$

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Optimum Unconstrained Partitions

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The *Dilworth truncation* f^D of a set function $f : 2^E \mapsto \mathbb{R}^N$ is the set function $f^D : 2^E \mapsto \mathbb{R}^N$ defined by

$$f^D(A) = \begin{cases} \min_{\mathbf{P} \in \Pi(A)} f(\mathbf{P}) & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases}$$

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Set partitioning formulation: w.l.o.g., assume $A = E$

Let $x_S = \begin{cases} 1 & \text{if } S \in \mathbf{P}; \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} f^D(E) = \quad \min \quad & \textstyle\sum_{S \subseteq E : S \neq \emptyset} f(S)\, x_S \\ \text{s.t.} \quad & \textstyle\sum_{S \subseteq E : j \in S} x_S = 1 \quad \forall j \in E \\ & x \geq 0 \\ & x \text{ integer} \end{aligned}$$

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LP relaxation:

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(D) & \max & \displaystyle\sum_{j\in E} y_j \\
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- except that here we may have $f(\emptyset) < 0$

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Hence we now consider the general case where we make no sign restriction on $f(\emptyset)$

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Given polyhedron $P \subseteq \mathbb{R}^E$ and $w \in \mathbb{R}^E$, assume w.l.o.g. that $E = \{e_1, \ldots, e_n\}$ with $w_{e_1} \geq w_{e_2} \geq \cdots \geq w_{e_n} \geq 0$

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Recursively define $y^G \in \mathbb{R}^E$ as follows

- for $j = 1, \ldots, n$ let
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If $P = \tilde{P}(f) = \{y \in \mathbb{R}^E : y(S) \leq f(S)\ \forall S \subseteq E,\ S \neq \emptyset\}$ for some set function f, then the Greedy Principle simplifies to:

- let $y^G(e_1) = f(\{e_1\})$ and for $j = 2, \ldots, n$ let

$$y^G_{e_j} = \min\left\{f(A + e_j) - y^G(A) : A \subseteq e_j^{\prec}\right\} \tag{1}$$

where $e_j^{\prec} = \{g \in A : g \prec e_j\} = \{e_1, \ldots, e_{j-1}\}$ for all $j = 1, \ldots, n$

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(i.e., optimum subset $A = e_j^{\prec}$)

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hence all these inequalities must hold as equalities □

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Hence *both y^G and x^G are optimal*, answering both Optimality Questions, and giving an efficient construction of an *optimum partition $\mathbf{P} = (P_1, \ldots, P_k)$*

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Therefore, for any subset $S \subseteq E$, the *value* $f^D(S)$ *of the Dilworth truncation* can be obtained in polynomial time, by solving $|S| - 1$ submodular minimization problems

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Therefore, for any subset $S \subseteq E$, the *value $f^D(S)$ of the Dilworth truncation* can be obtained in polynomial time, by solving $|S| - 1$ submodular minimization problems

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- If $S = \emptyset$ then $f^D(u + v) \leq f^D(u) + f^D(v)$ (Why?)

Submodularity of the Dilworth Truncation (continued)

Else, i.e., $S \neq \emptyset$, number the elements in E so $S = e^{\prec}_{i+1}$, $e_{i+1} = u$ and $e_{i+2} = v$

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QED

Asymptotics of Potts Partition Functions
(Anglès d'Auriac & al., 2002)

Statistical Mechanics	*Graph Theory*
Lattice (V, E)	Graph $G = (V, E)$
Site $i \in V$	Node
Bond $ij \in E$	Edge
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An Application in Statistical Mechanics

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Energy of *configuration* $\sigma = (\sigma_1, \ldots, \sigma_n)$: $\mathbf{E}(\sigma) = \sum_{ij \in E} K_{ij} \delta_{\sigma_i \sigma_j}$

where the *Kronecker symbol* $\delta_{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$

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Let $\alpha_{ij} = \log_q \nu_{ij}$ so $Z(K) = \sum_{F \in 2^E} q^{h(F)}$
where $h(F) = nc(F) + \sum_{ij \in F} \alpha_{ij}$

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When q goes to infinity, $Z(K) \to Nq^{h^*}$ where N is the number of optimum sets F and

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- Can we do better than general SFMin?

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   then we may add to F^* all edges in E within each P_i

Therefore $h(F^*) = \alpha(E) - \sum_{i=1}^{k} f(P_i)$

where $f : 2^V \mapsto \mathbb{R}$, defined by $f(S) = \frac{1}{2}\left(\sum_{j \in S, k \notin S} \alpha_{jk}\right) - 1$,

is the cut function of the graph $G = (V, E)$ with edge "capacities" $\alpha \geq 0$, *minus the constant 1*

- so, $f(\emptyset) = -1 < 0$

A Faster Algorithm

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The minimizations at each step of the Greedy Algorithm can be performed efficiently by *network flow techniques* (*minimum s, t-cuts* in an associated network)

The running time is O($|V|^2\,|E|$)

- much faster than general SFMin on the old ground set $|E|$

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The function g_f defined by $g_f(S) = f(S) + f(E \setminus S)$ is:

- symmetric; and
- *submodular if f is submodular*

Sym-SFMin

If g is symmetric and submodular then, for all $S \subseteq E$

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The Optimum Bipartition problem with submodular part costs, is equivalent to the **Symmetric Submodular Minimization problem (Sym-SFMin)**:

- given a symmetric submodular function $g : 2^E \mapsto \mathbb{R}$
- find a *proper* subset S of E which minimizes $g(S)$

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Sym-SFMin and Decomposition

Proposition Assume that f is normalized and submodular, and that there exists a proper subset $A \subset E$ such that $\breve{f}(A) = f(A) + f(\bar{A}) - f(E)$ satisfies $\breve{f}(A) = 0 = \breve{f}(\emptyset) = \breve{f}(E)$. where $\bar{A} = E \setminus A$. Then f is decomposable as

$$f(B) = f(B \cap A) + f(B \cap \bar{A}) \quad \text{for all } B \subseteq E$$

Proof: Since f is normalized and submodular

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Separators

A proper subset A of E such that

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- Hence, the separators partition E

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- *either* S^* separates u and v, and we may choose $S^* = \{u\}$
- *or else* u and v are on the same side of S^* and we may *contract* u and v into a single element

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- the ground set E with $E_{u,v} = (E - u - v) + uv$
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Corollary: If f is submodular, then (a_n, a_{n-1}) is a pendent pair for its symmetric function g_f

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- symmetric submodular function subject to hereditary family constraints (Goemans & Soto, 2013): $\min\{f(S) : S \in \mathcal{I}\}$ where $\mathcal{I} \subseteq 2^V$ satisfies, for all $A, B \subseteq V$,
  $\emptyset \neq A \subset B \in \mathcal{I} \quad \Rightarrow \quad A \in \mathcal{I}$

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$\Rightarrow$ When f is submodular, O(n^4) EO's suffice

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- . . . see Thursday afternoon talk for related complexity results and open questions

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Theorem: [Q 1999; Zhao, Nagamochi & Ibaraki 2005]
If g is symmetric, submodular and nonnegative, then (for every $k \geq 2$) the Greedy Splitting Algorithm produces a k-way partition with total cost at most $2 - \frac{2}{k}$ times the optimum

Notes

Short Course on Submodular Functions
Part 2: Extensions and Related Problems
Session 2.B: SFmax

S. Thomas McCormick   Maurice Queyranne

Sauder School of Business, UBC

JPOC Summer School, June 2013

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- cannot be approximated within a ratio better (larger) than $1 - 1/e \approx 0.632$, unless P = NP (Feige 1998)

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Starting with $S_0 = \emptyset$, repeat the following greedy step:

for $i = 0, \ldots, (k-1)$ let

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- and therefore $f(S_k) \geq \left(1 - \frac{1}{e}\right)\mathrm{OPT}_k > 0.632\,\mathrm{OPT}_k$

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- typical practical performance is much better

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Proof of Greedy Performance

- Fix ℓ (size of greedy solution) and k (size of optimal set)
- Fix $S^* \in \arg\max\{f(S) : S \subseteq E,\ |S| \leq k\}$, so $f(S^*) = \mathsf{OPT}_k$
- Assume w.l.o.g., that $|S^*| = k$ and let $S^* = \{v_1^*, \ldots, v_k^*\}$
- Let $(v_1, \ldots, v_\ell)$ be the greedy order chosen by the algorithm
- Then, for all $i < \ell$, the gap

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$\delta_\ell \ \leq\ (1 - \frac{1}{k})^\ell\, \delta_0 \ \leq\ (1 - \frac{1}{k})^\ell\, \mathsf{OPT}_k \ \leq\ e^{-\ell/k}\, \mathsf{OPT}_k$ QED

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Idea: to reduce the number of function evaluations and of comparisons, store upper bounds α_v on the increments $f(v|S_i)$ in a priority queue, and only update α_v when element v is examined

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In practice, Minoux's trick often yields enormous speedups (over 700-fold) over standard implementation of Greedy, for very large data sets

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  - and based on local search (not on a greedy approach!)

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- in fact, most of these problems are PLS-complete

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Modified Local Search (**MLS**): given $\epsilon > 0$,

1. Initialization: find a (feasible) solution S
2. While there exists an ϵ-improving solution $S^+ \in N(S)$ do $S := S^+$
3. Output S

If $f(S_0) > 0$ then after k iterations the current solution S_k satisfies $f(S_k) > (1+\epsilon)^k f(S_0)$
$\Rightarrow$ If $\log(\text{OPT}/f(S_0))$ is polynomially bounded (in the instance input size) then for every fixed $\epsilon > 0$, MLS terminates and outputs an ϵ-local optimum after at most $\log(\text{OPT}/f(S_0)) \,/\, \log(1+\epsilon)$ iterations, i.e. in polytime

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$$f(R) \quad \leq \quad f(R + u) + f(S - u) - f(S)$$

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The proof of 2 is similar QED

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Other recent approximation results for monotone and non-monotone SFMax subject to a variety of constraints

- one or several knapsacks, matroidal constraints, ...

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Short Course on Submodular Functions
Part 2: Extensions and Related Problems
Session 3: Submodularity in Vector Spaces

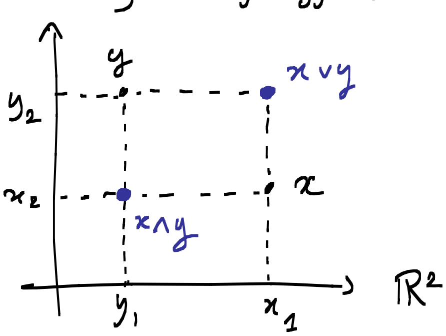
S. Thomas McCormick and Maurice Queyranne
Sauder School of Business, UBC

Rappels : Un **treillis** (en anglais : *lattice*) est un un ensemble partiellement ordonné (un "*poset*") $(L, \leq)$ tel que, pour tous $a, b \in L$ il existe

- une plus petite borne supérieure commune $a \vee b$, le **supremum** (ou **sup**) de a et b (en anglais, *the join of a and b*), c'est à dire un unique élément $s = a \vee b \in L$ tel que $a \leq s$, $b \leq s$ et pour tout $c \in L$ tel que $a \leq c$ et $b \leq c$ on doit avoir $s \leq c$
- et une plus grande borne inférieure commune $a \wedge b$, (l'**infimum** (ou **inf**) de a et b (*the meet of a and b*) : $a \wedge b \leq a$, $a \wedge b \leq b$ et $\forall c \in L$ $(c \leq a$ et $c \leq b) \Rightarrow c \leq a \wedge b$

Exemples : $(2^E, \subseteq)$ avec $a \vee b = a \cup b$ (union) et $a \wedge b = a \cap b$ (intersection)

- $(\mathbb{T}^E, \Rightarrow)$ où $\mathbb{T} = (\text{Vrai}, \text{Faux})$, $\mathbb{T}^E$ est l'ensemble des fonctions logiques définies sur E,
  $\Rightarrow$ est l'implication ($a \Rightarrow b$ signifie que $b(e) = \text{Vrai}$ pour tous les $e \in E$ tels que $a(e) = \text{Vrai}$)
  $\vee$ est la disjonction ("ou" logique : $\forall e \in E$ $a \vee b(e) = \text{Vrai}$ si et seulement si l'un au moins de $a(e)$ ou $b(e) = \text{Vrai}$)
  $\wedge$ est la conjonction ("et" logique : $\forall e \in E$ $a \wedge b(e) = \text{Vrai}$ ssi $a(e) = b(e) = \text{Vrai}$)
- $(\mathbb{R}^d, \leq)$ où $\leq$ est l'ordre partiel "par composantes" $x \leq y \Leftrightarrow x_j \leq y_j$ $\forall j = 1..d$
  $\vee$ est le supremum par composantes
  $(x \vee y)_j = x_j \vee y_j = \max\{x_j, y_j\}$ $\forall j = 1..d$
  $\wedge$ est l'infimum par composantes
  $(x \wedge y)_j = x_j \wedge y_j = \min\{x_j, y_j\}$ $\forall j = 1..d$



On définit de même $(\mathbb{Z}^d, \leq)$, $(\mathbb{B}^d, \leq)$ (où $\mathbb{B} = \{0,1\}$)
et plus généralement $(\bigotimes_{j=1}^{d} A_j, \leq)$ où $\bigotimes_{j=1}^{d} A_j$ est le produit Cartésien de sous-ensembles arbitraires $A_j \subseteq \mathbb{R}$, avec l'ordre partiel par composantes $\leq$

En particulier, pour $\ell, u \in \mathbb{Z}^d$ tels que $\ell \leq u$,
la **boîte** $B_{\ell,u} = \{x \in \mathbb{Z}^d : \ell \leq x \leq u\}$ est un treillis
(c'est un **sous-treillis** de $\mathbb{Z}^d$, c'est à dire un sous-ensemble de $\mathbb{Z}^d$ stable (ou fermé) pour les opérations $\vee$ et $\wedge$ de $(\mathbb{Z}^d, \leq)$)

Remarques: 1) on définit de même les boîtes (ou rectangles) dans $\mathbb{R}^d$
2) les treillis $(2^E, \subseteq)$, $(\mathbb{T}^E, \Rightarrow)$ et $(\mathbb{B}^E, \leq)$ sont, naturellement, "isomorphes"
3) les sous-treillis de $(2^E, \subseteq)$ sont les anneaux d'ensemble
4) dans tout treillis, on a les équivalences $a \leq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a$

Fonctions sous-modulaires dans les treillis et les espaces vectoriels :

Une fonction $f : L \to \mathbb{R}$ est **sous-modulaire** ssi

$$f(a \vee b) + f(a \wedge b) \leq f(a) + f(b) \qquad \forall a, b \in L$$

Caractérisations de la sous-modularité

- dans $\mathbb{Z}^d$: si $(L, \leq)$ est un sous treillis de $\mathbb{Z}^d$, $f : L \to \mathbb{R}$ est sous-modulaire ssi elle satisfait la **propriété d'incréments décroissants** :

$$f(x+e_i+e_j) - f(x+e_j) \leq f(x+e_i) - f(x) \qquad \forall x \text{ tel que } x+e_i \text{ et } x+e_j \in L$$

où $e_i = (0,\dots,0,\underset{\uparrow\, i^e \text{ position}}{1},0\dots,0)^T$ est le i^{eme} vecteur unitaire

exercice : prouver cette équivalence

- dans $\mathbb{R}^d$: $f : \mathbb{R}^d \to \mathbb{R}$ différentiable est sous-modulaire

$$\Leftrightarrow \quad \frac{\partial}{\partial x_i} f(x) \text{ est non-croissante en } x_j \quad \forall i \neq j$$

$$\Leftrightarrow \quad \frac{\partial^2}{\partial x_i \partial x_j} f(x) \leq 0 \quad \forall i \neq j$$

exercice : prouver cette équivalence

Remarque : cette dernière condition montre que la sous-modularité est différente à la fois de la convexité et de la concavité : en effet $f : \mathbb{R}^d \to \mathbb{R}$, deux fois différentiable, est

- sous modulaire ssi son Hessien $Hf(x) = \left(\frac{\partial^2}{\partial x_i \partial x_j} f(x)\right)_{\substack{i=1..d \\ j=1..d}}$ a, pour tout $x \in \mathbb{R}^d$, tous ses termes non-diagonaux qui sont non-positifs. (une propriété indépendante des termes diagonaux $\frac{\partial^2}{\partial x_i^2} f(x)$)
- convexe ssi, $\forall x \in \mathbb{R}^d$, $Hf(x)$ est positif semi-défini (psd)
- concave ssi, $\forall x \in \mathbb{R}^d$, $-Hf(x)$ est psd

(propriétés de toute la matrice $Hf(x)$)

SFMin dans une boîte discrète

étant donnés $l \leq u \in \mathbb{Z}^d$

et $f : B_{l,u} \to \mathbb{Q}$ sous-modulaire, donnée par un oracle de valeur

$$\text{SFMin}(B_{l,u}) : \quad \min \{ f(x) : x \in \mathbb{Z}^d,\ l \leq x \leq u \}$$

- Peut-on résoudre ce problème en temps polynomial (polynomial en d, les tailles d'input de l et u et d'une borne supérieure $M \geq \max\{|f(x)| : x \in B_{l,u}\}$) ?
- La réponse est NON :

Proposition : Tout algorithme pour résoudre $\text{SFMin}(B_{l,u})$ doit utiliser au moins $\sum_{i=1}^{d} (u_i - l_i + 1)$, un nombre pseudo-polynomial, d'appels à l'oracle de valeur.

Preuve : Toute fonction séparable $f = \sum_{i=1}^{d} f_i$ définie sur $B_{l,u}$, c.a.d., $f(x) = \sum_{i=1}^{d} f_i(x_i)$ où chaque $f_i : \{l_i, l_i+1, .., u_i\} \to \mathbb{Q}$ est sous-modulaire

exercice : vérifier cette affirmation

Comme les fonctions f_i peuvent être quelconque, il faut connaître toutes leurs valeurs pour pouvoir en minimiser la somme.

[Plus précisément, on définit la stratégie adverse suivante pour l'oracle de valeur : retourner la valeur $f(x) = d$ pour toute requête $x \in B_{\ell,u}$.
Alors, pour toute séquence de moins de $\sum_{i=1}^{d} f_i(x_i)$ requêtes il existe une coordonnée i et une valeur $v_i \in \{\ell_i, \ell_{i+1}, \dots, u_i\}$ qui n'apparait dans aucune requête. L'algorithme est incapable de différencier les fonctions $f^1 = f_i^1 + \sum_{j \neq i} f_j$ et $f^2 = f_i^2 + \sum_{j \neq i} f_j$ où $f_j(v) = 1$ pour toutes les coordonnées $j = 1..d$ et valeurs v, sauf que $f_i^1(v_i) = 0$ et $f_i^2(v_i) = 2$, et

$$\operatorname{argmin}\{f^1(x) : x \in B_{\ell,u}\} = \{x \in B_{\ell,u} : x_i = v_i\} \text{ alors que}$$

$$\operatorname{argmin}\{f^2(x) : x \in B_{\ell,u}\} = \{x \in B_{\ell,u} : x_i \neq v_i\}$$]

QED

Remarque : cet argument implique aussi une borne supérieure de $(1 - \frac{1}{d-1})$ sur l'approximabilité de SFMin($B_{\ell u}$) lorsque $f \geq 0$

Voici un **algorithme pseudo-polynomial pour SFMin($B_{\ell u}$) :**

- on définit l'**expansion unaire** de chaque coordonnée :

$$x_j = \ell_j + \sum_{k=1}^{w_j} y_{j,k} \quad \text{où } w_j = u_j - \ell_j \text{ et chaque } y_{j,k} \in \mathbb{B} \text{ satisfait}$$

$$y_{j,1} \geq y_{j,2} \geq \dots \geq y_{j,w_j}$$

- soient $E = \{(j,k) : j = 1..d,\ k = 1..w_j\}$ l'ensemble des indices de ces variables $y_{j,k}$

$$\mathcal{F} = \{S \subseteq E : (j,k) \in S \Rightarrow (j,k-1) \in S \quad \forall j = 1..d,\ \forall k = 1 \dots w_j - 1\}$$

$\varphi : \mathcal{F} \to B_{\ell,u}$ où $x = \varphi^{-1}(S)$ a pour composantes

$$x_j = \ell_j + |\{k : (j,k) \in S\}|$$

$$F = f \circ \varphi : \mathcal{F} \to \mathbb{Q} \quad (\text{c.à.d, } F(S) = f(\varphi(S)))$$

exercice : vérifier que

- $\mathcal{F}^i$ est stable pour l'union et l'intersection, donc un anneau d'ensembles
- φ est une bijection, et $S \subseteq T \iff \varphi(S) \leq \varphi(T)$

  donc φ est un isomorphisme de (sous-) treillis
- F est une fonction sous-modulaire sur l'anneau d'ensembles $\mathcal{F}^i$

  et $x \in \operatorname{argmin}\{f : x \in B_{\ell,u}\} \iff \varphi^{-1}(x) \in \operatorname{argmin}\{F(S) : S \in \mathcal{F}^i\}$

On peut donc résoudre SFMin$(B_{\ell,u})$ en temps pseudo-polynomial en résolvant SFMin pour la fonction F sur l'anneau d'ensembles $\mathcal{F}^i$

Référence (dans la liste distribuée avec les Problèmes)

[22] K. Murota (2003) Discrete Convex Analysis (livre)