Manifolds, Room Partitionings, Abstract Sperner and PPAD

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Combinatorial manifolds

Given: **rank** r

collection M of r-element sets called **rooms**

set of **vertices** $V = \bigcup M$

wall = room without a vertex v (wall “opposite” v)

any wall belongs to exactly 2 rooms

(i.e. any $(r-1)$-set of vertices belongs to 0 or 2 rooms)

call M a manifold.

Room partitionings

Given a manifold M with vertex set V,

room partitioning = partition of V into rooms.

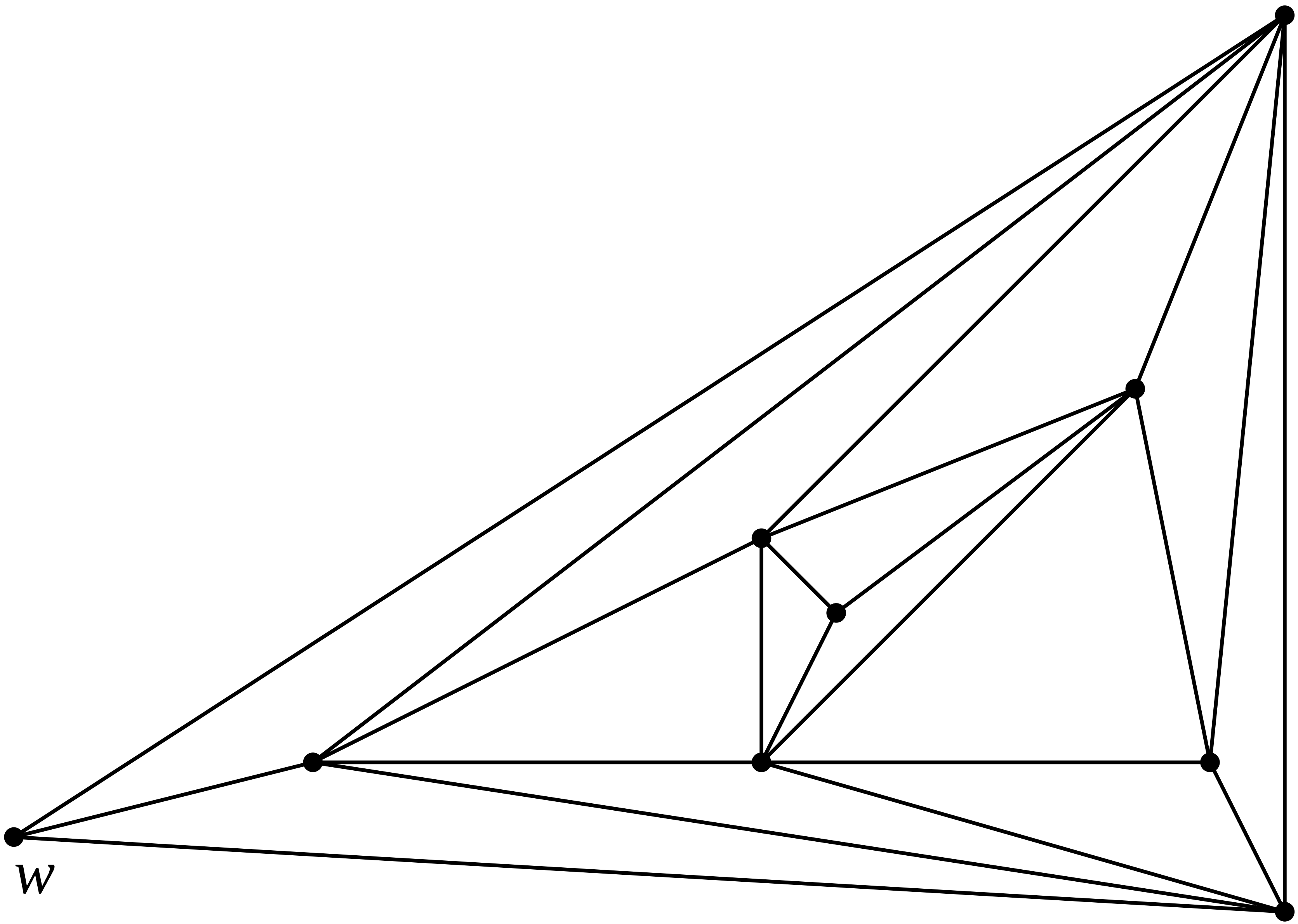
(Then |V| is a multiple of the rank r.)

Theorem.

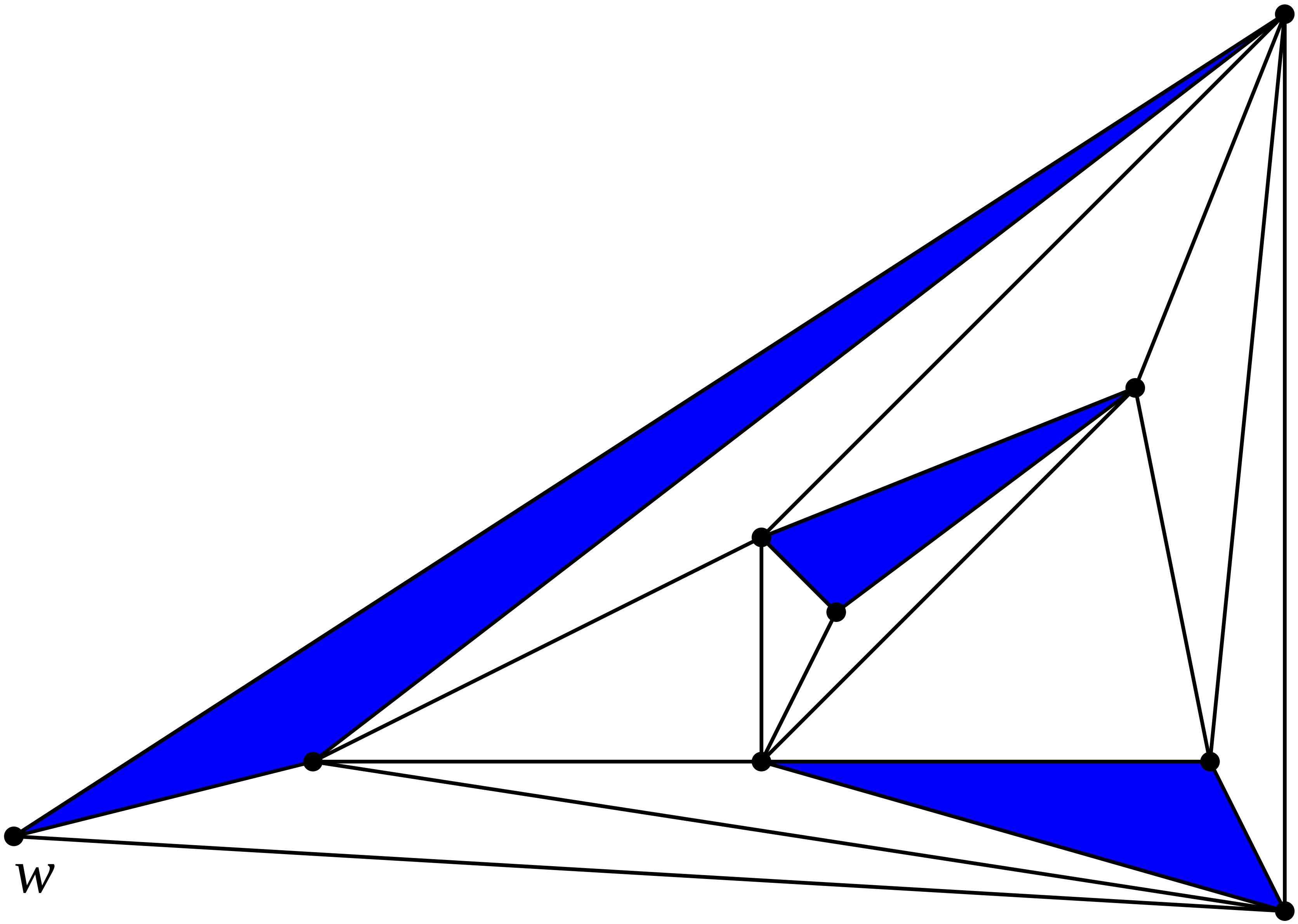
M has an even number of room partitionings.

Proof by Parity Argument (PPA).

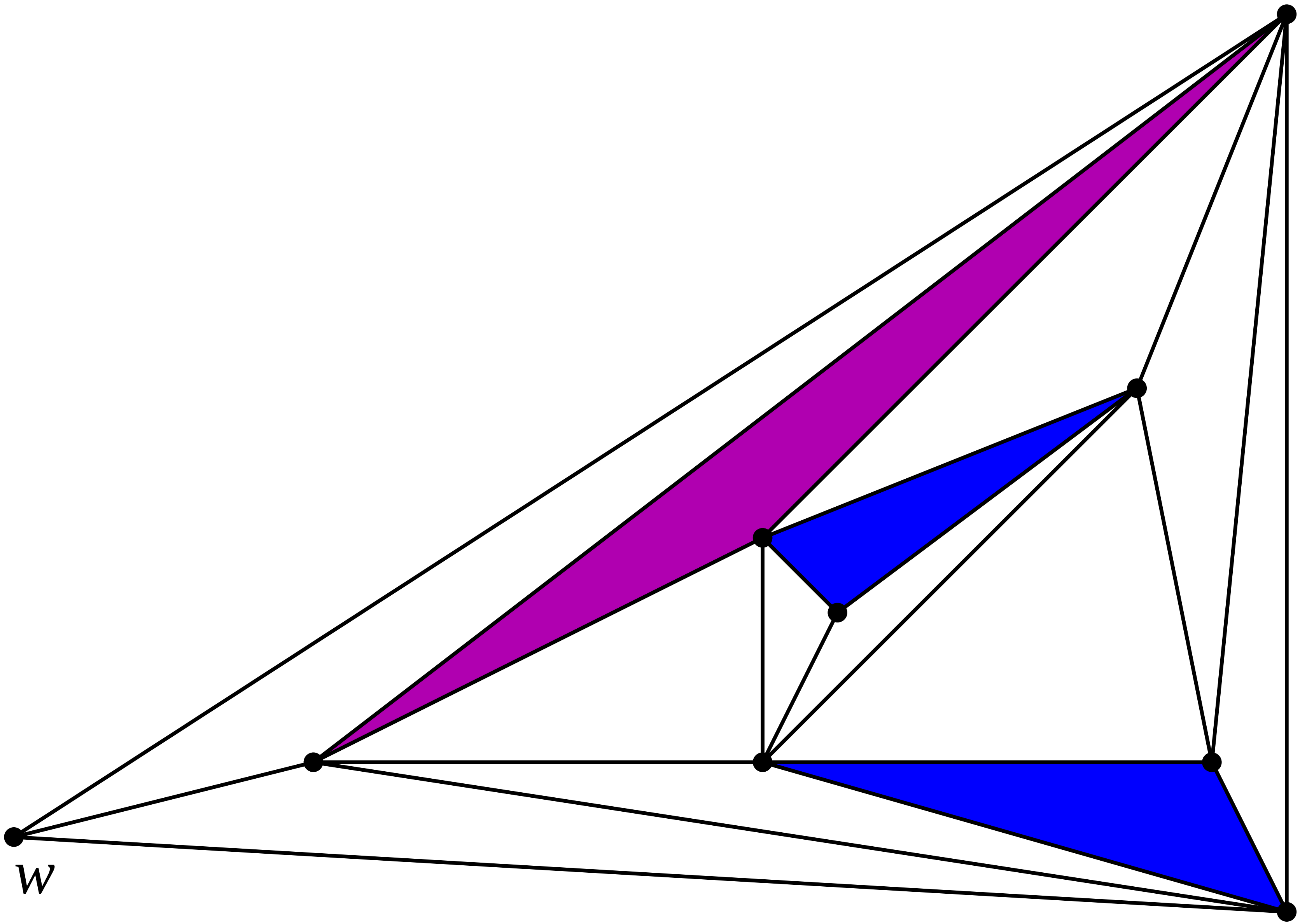
Example: Manifold



Room partitioning



w-almost room partitioning



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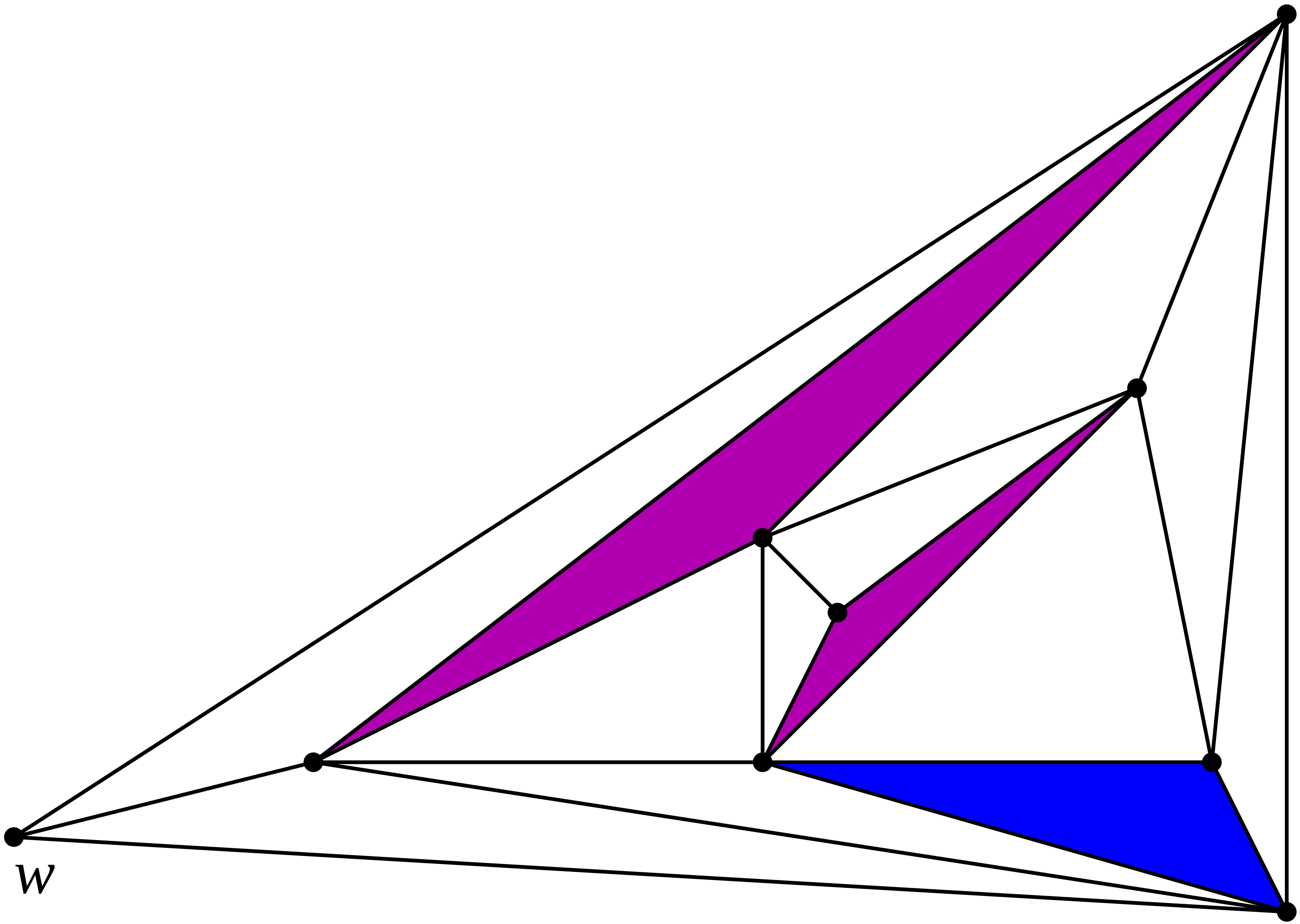

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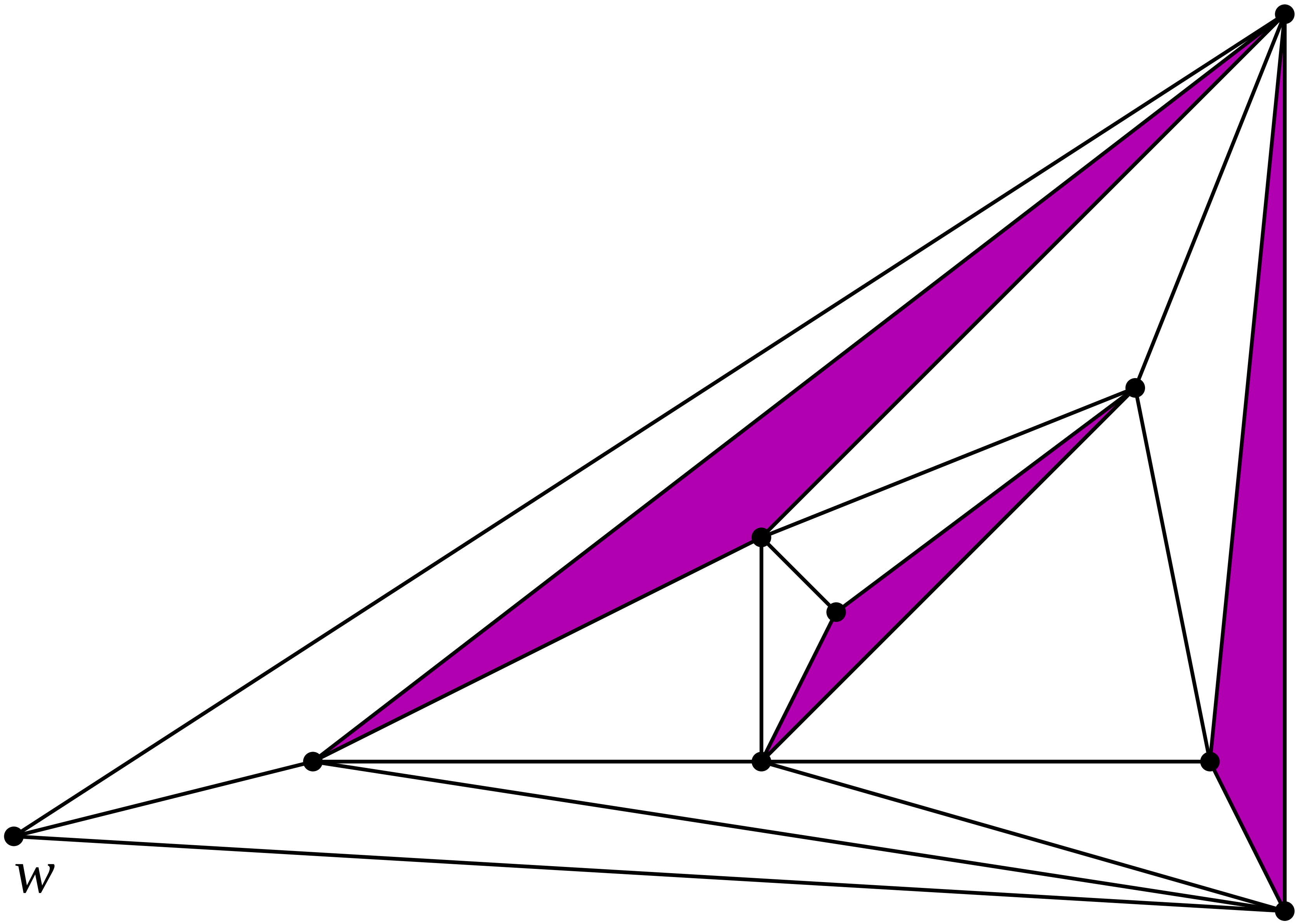

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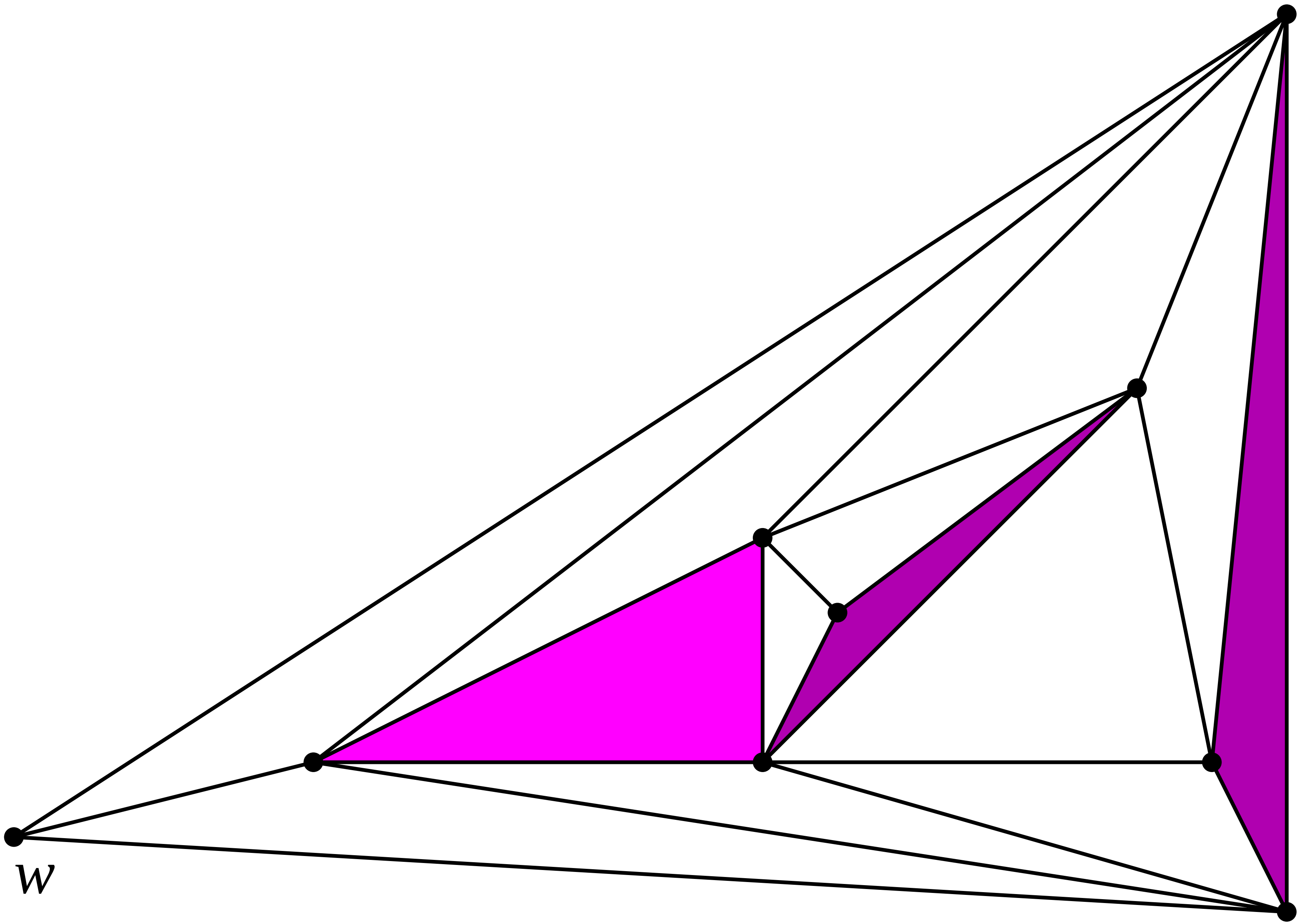

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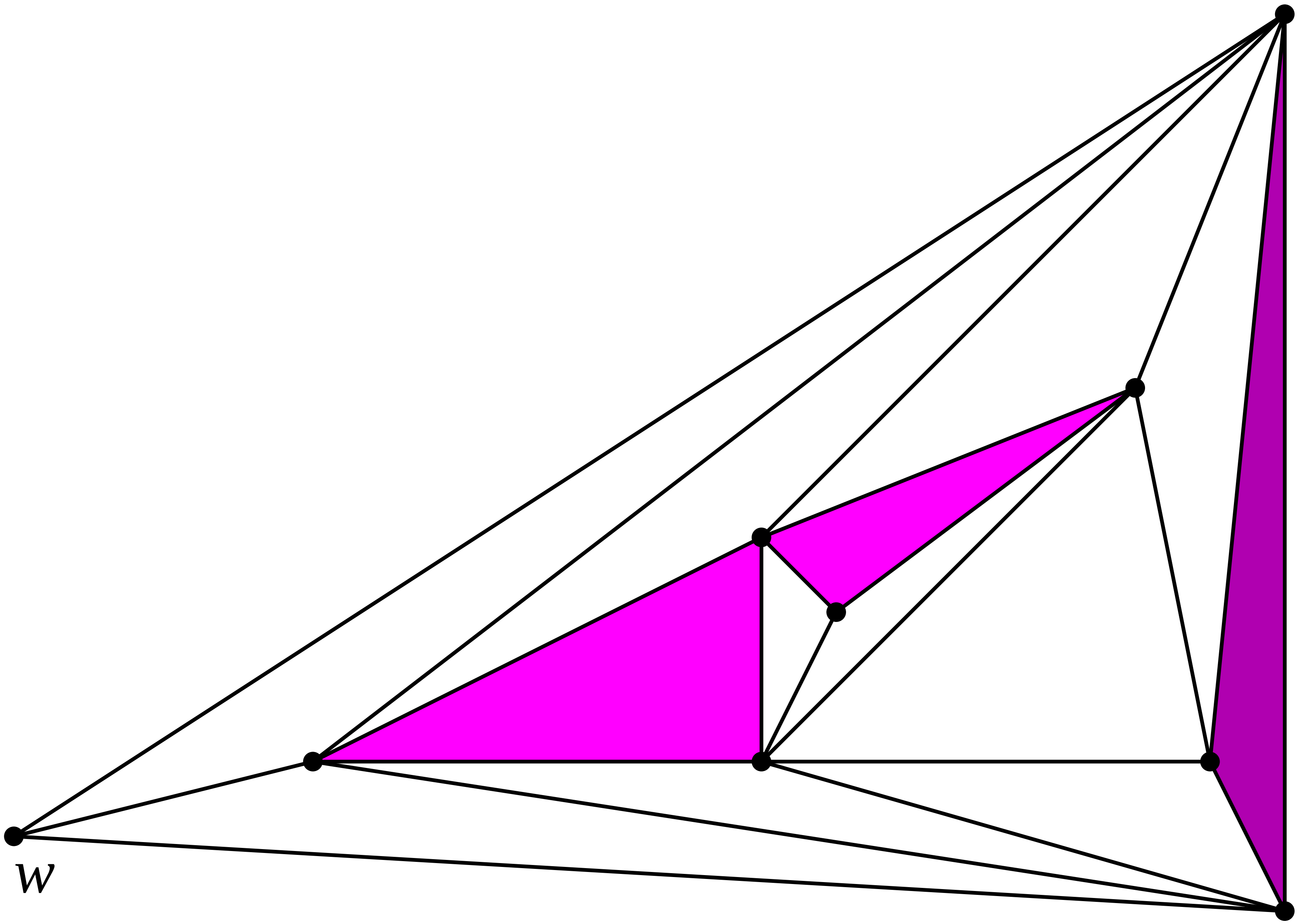

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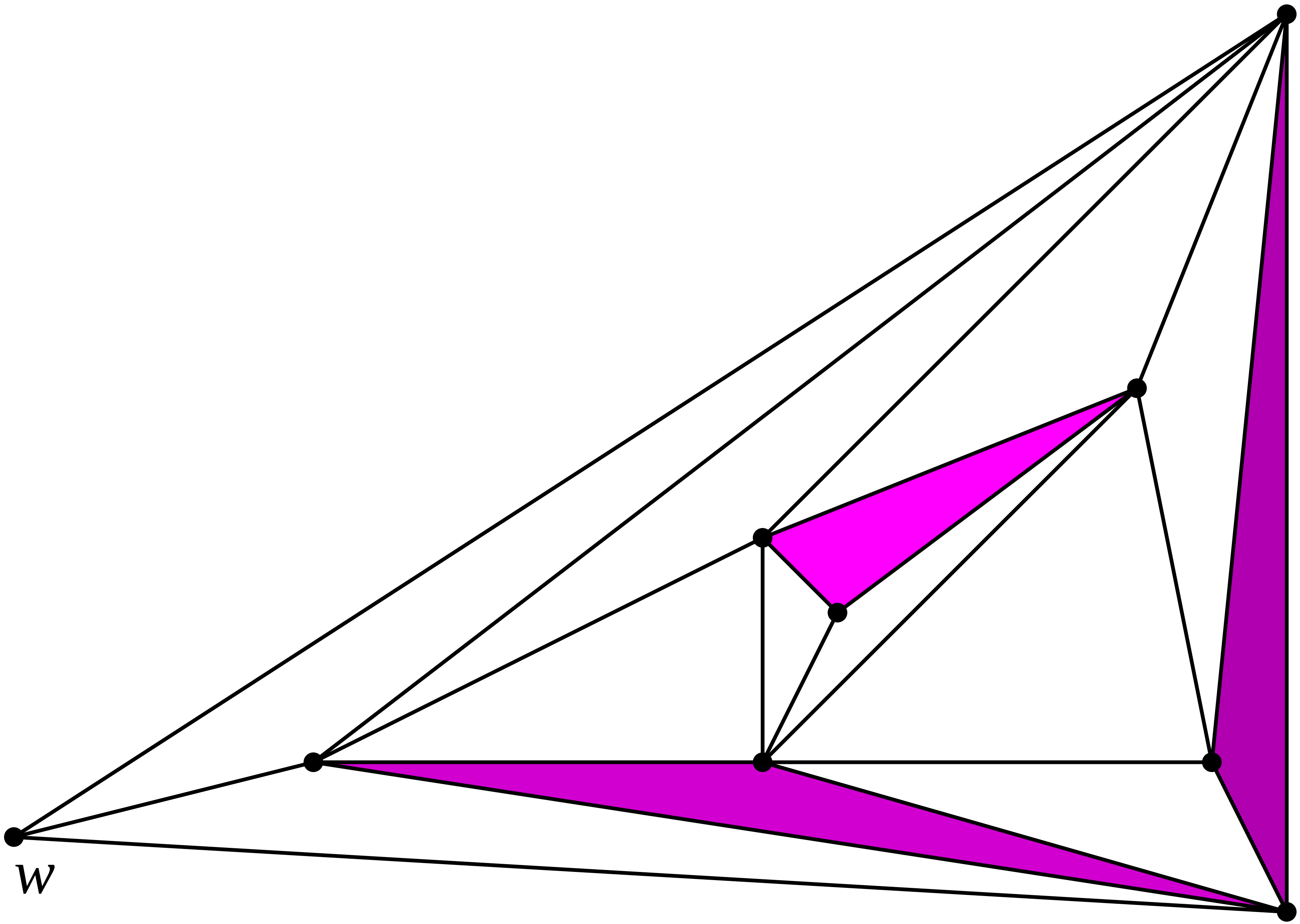

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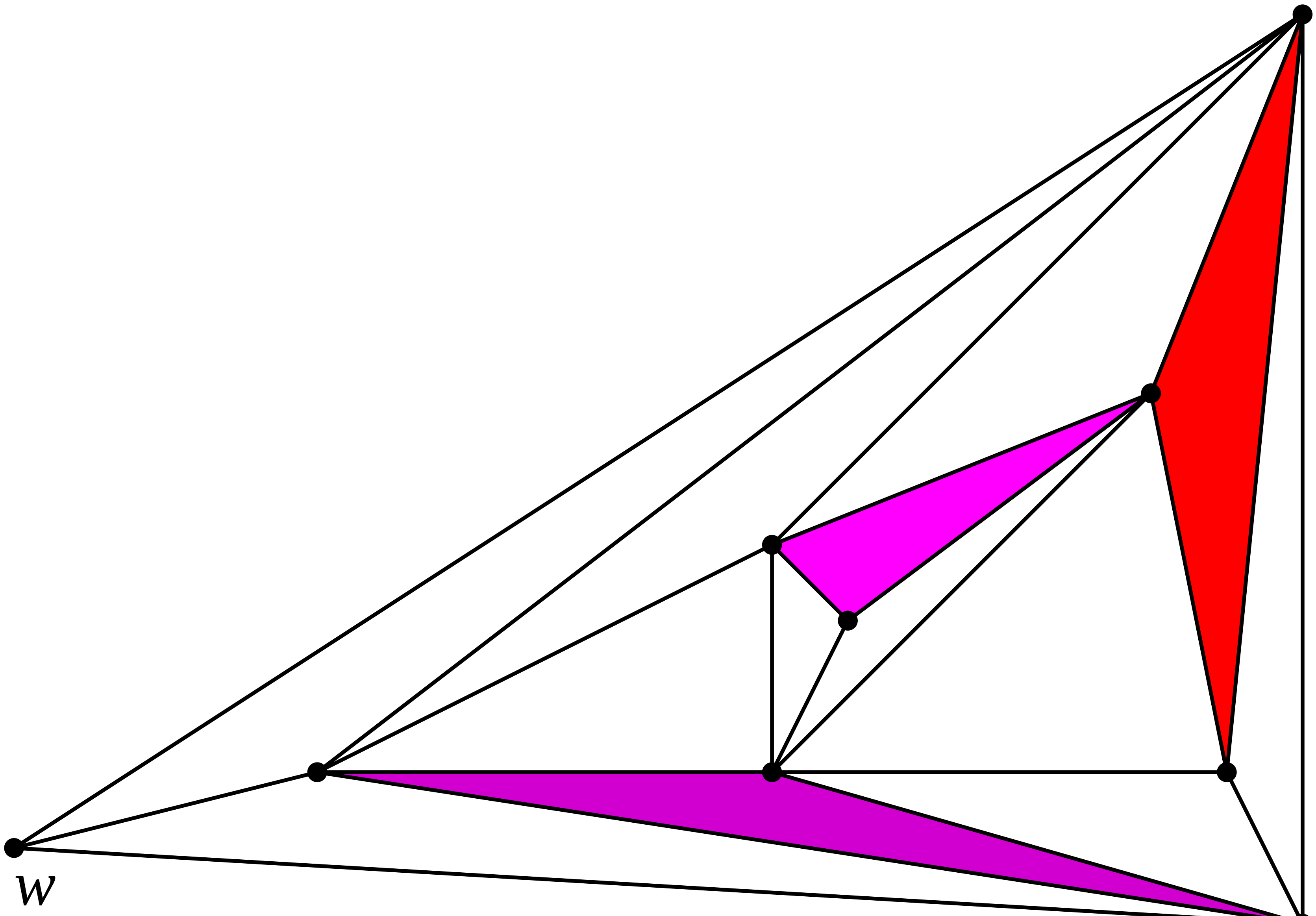

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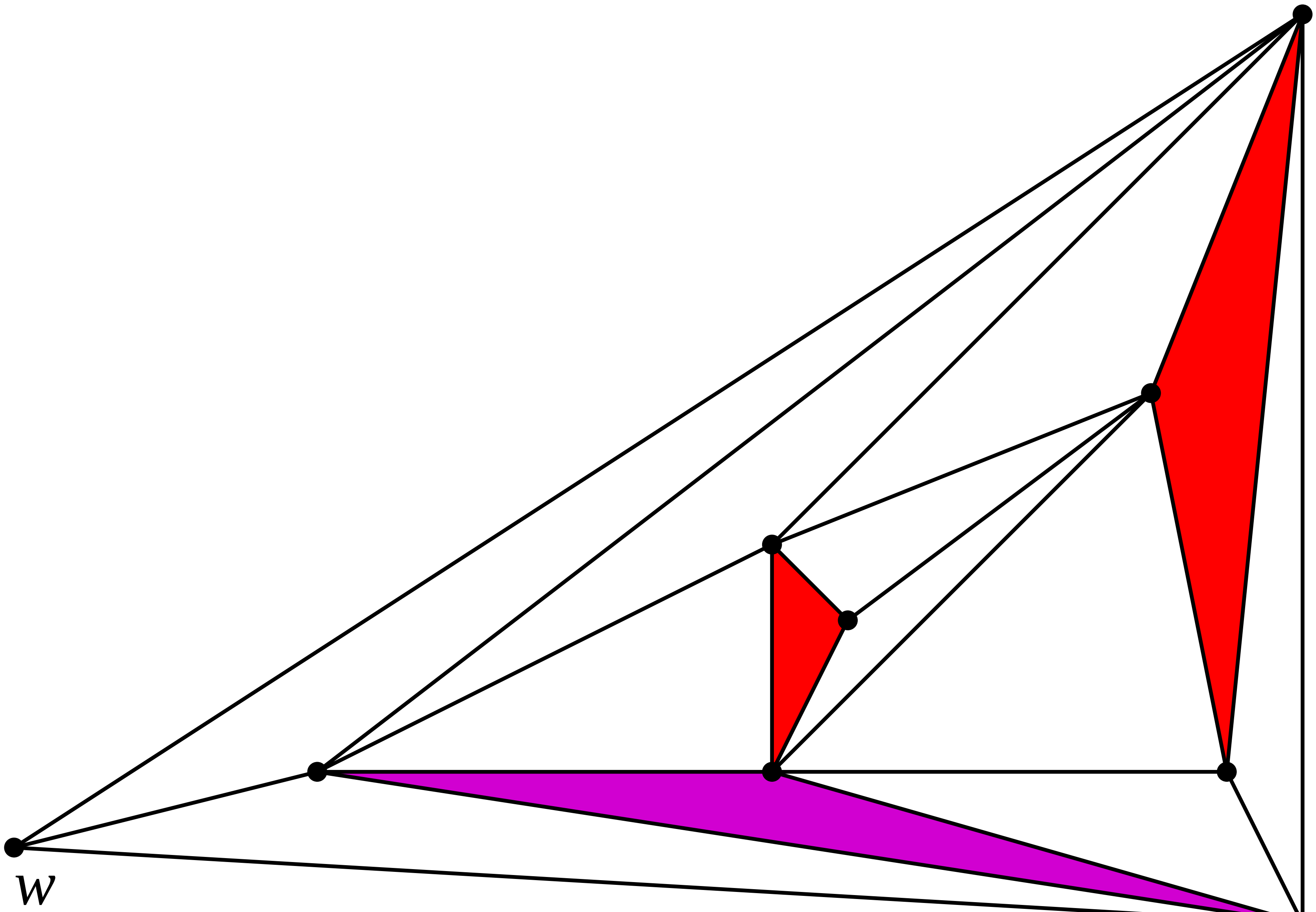
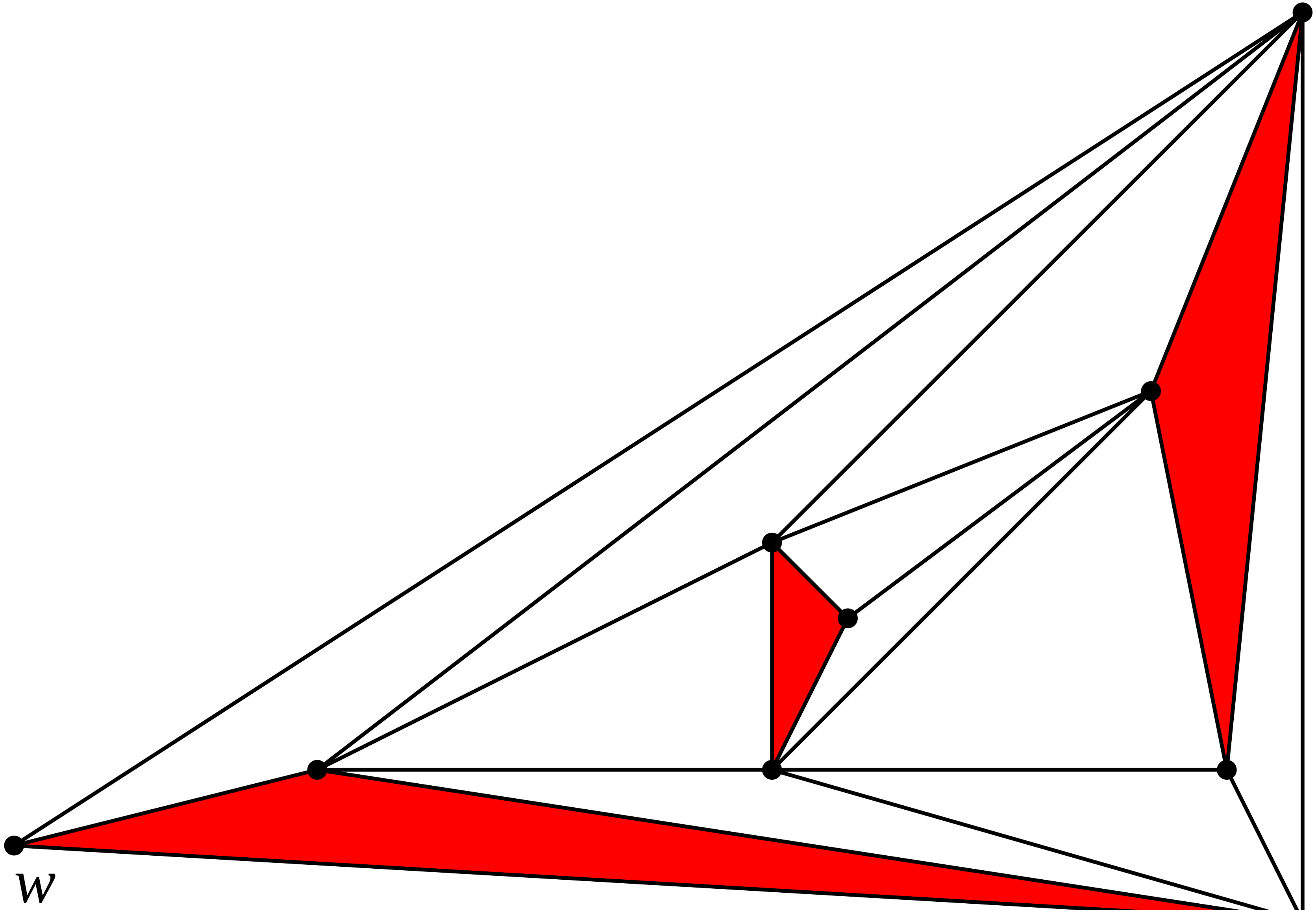
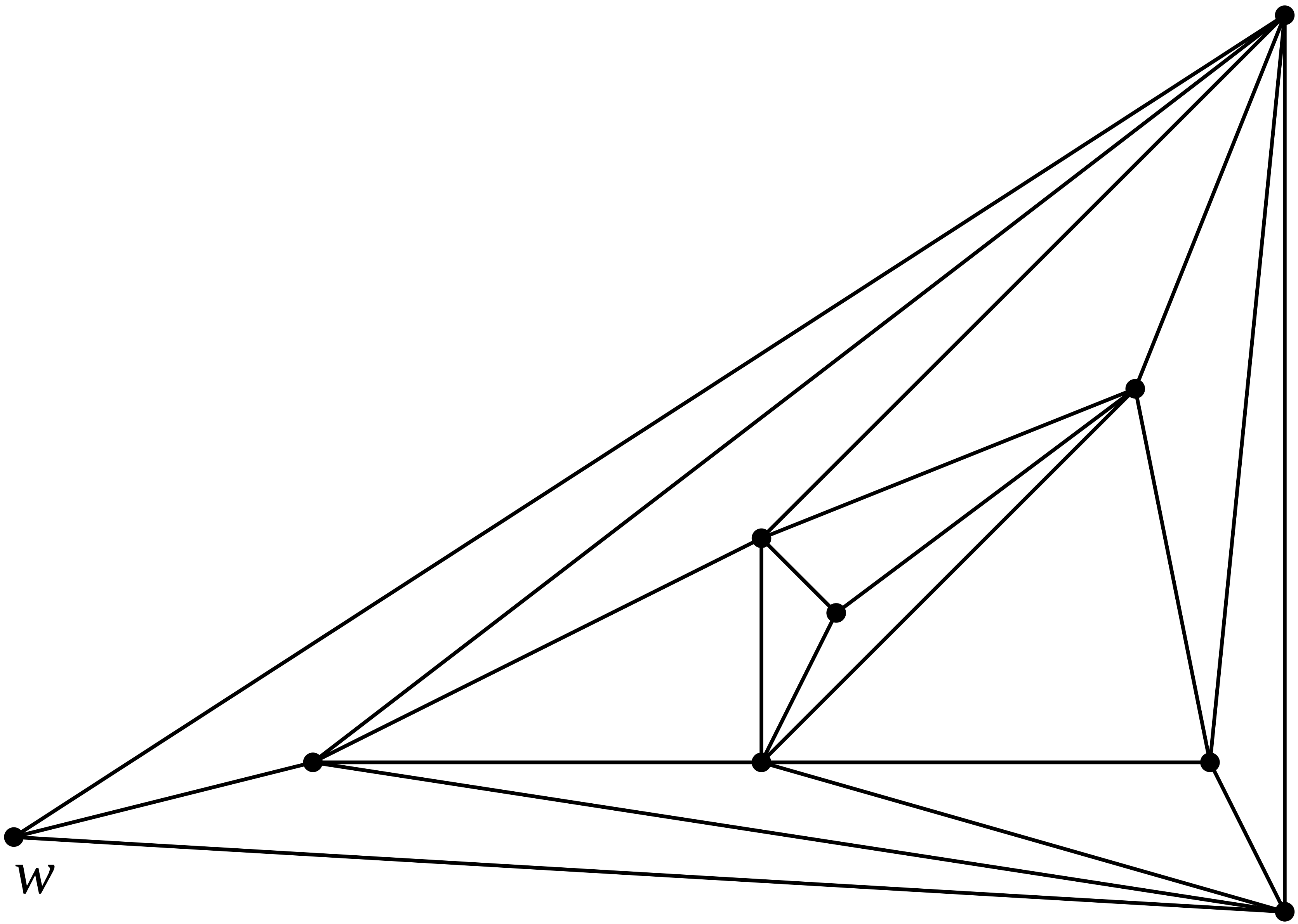


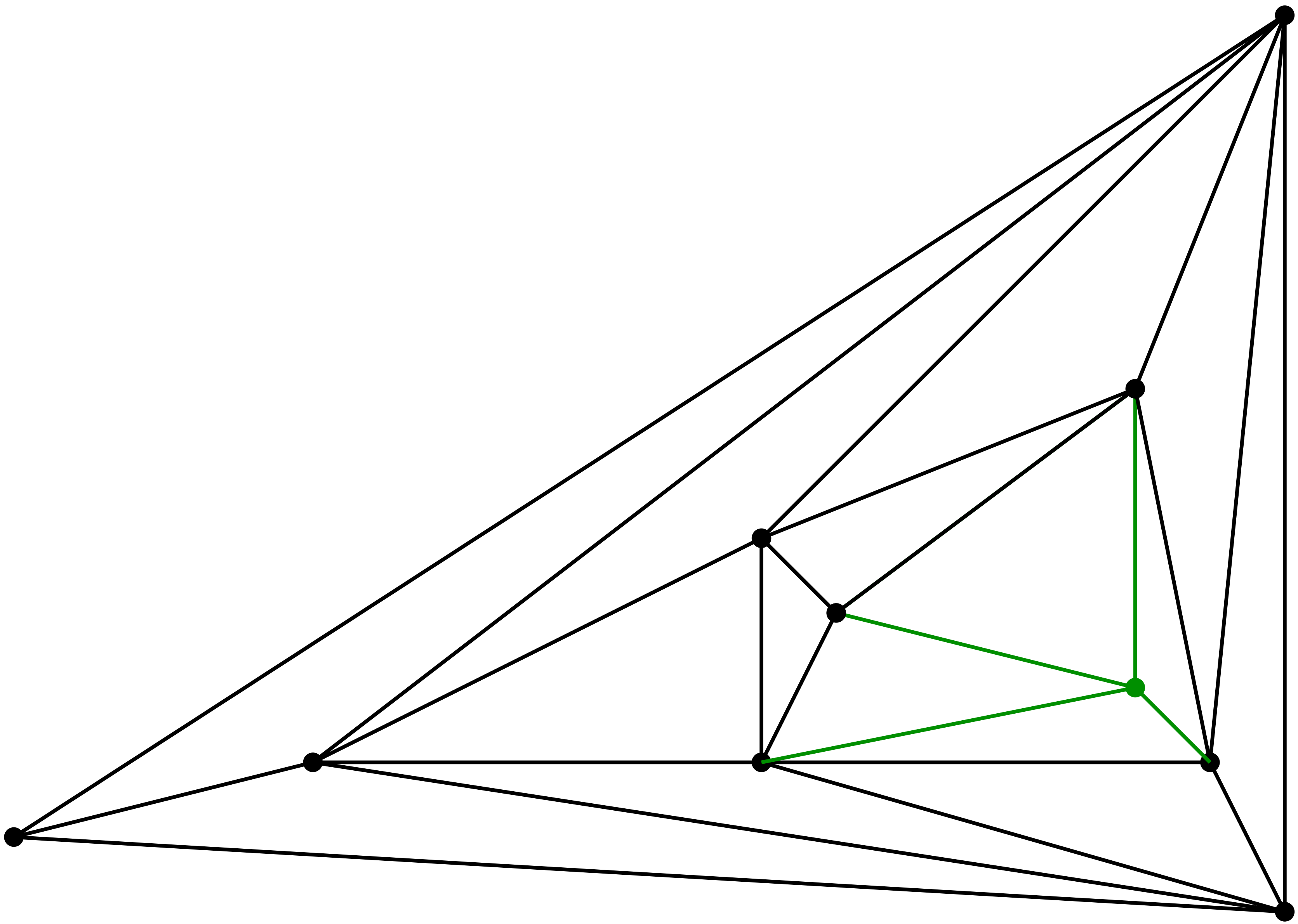
New room partitioning



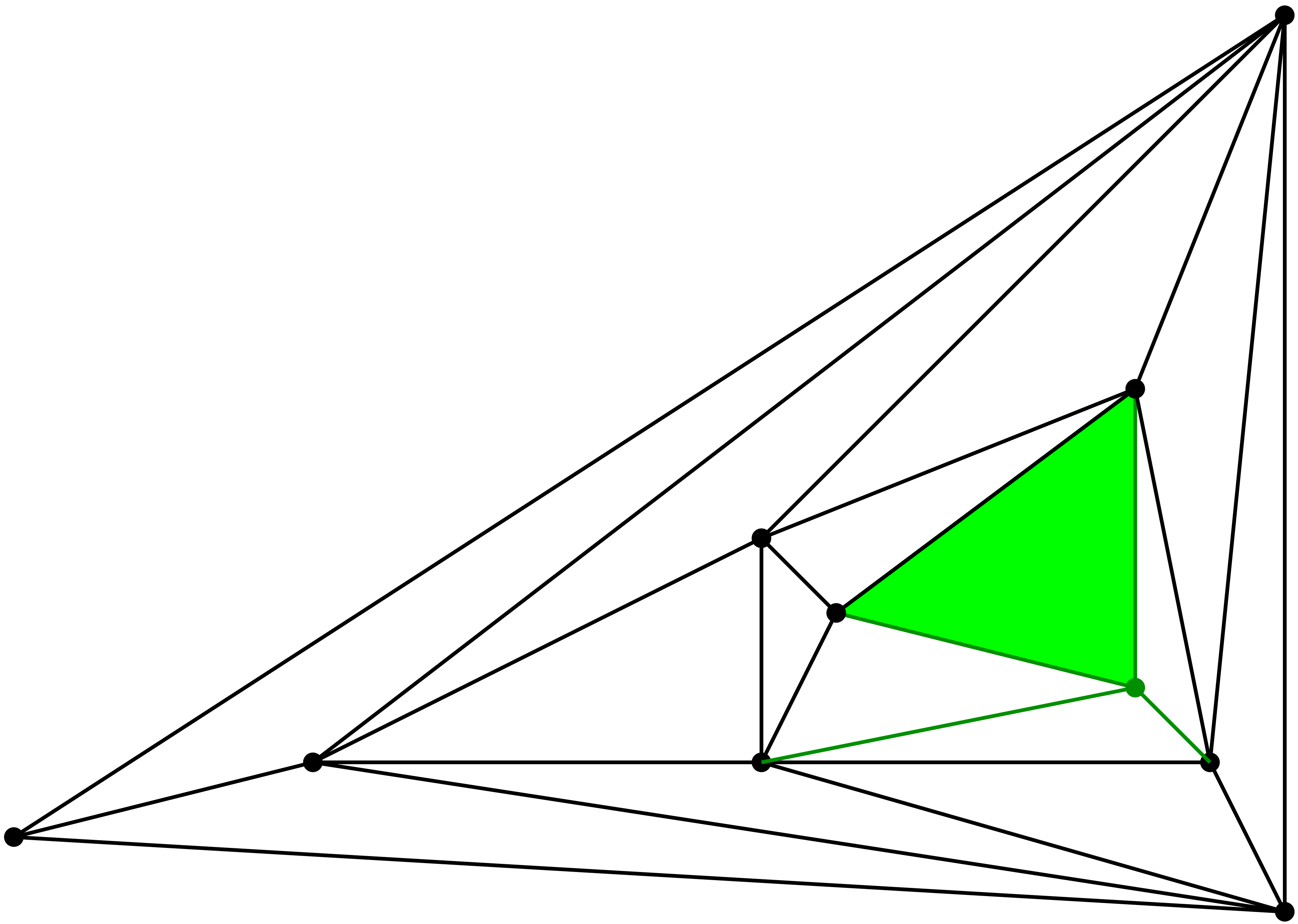
Construct exponential example



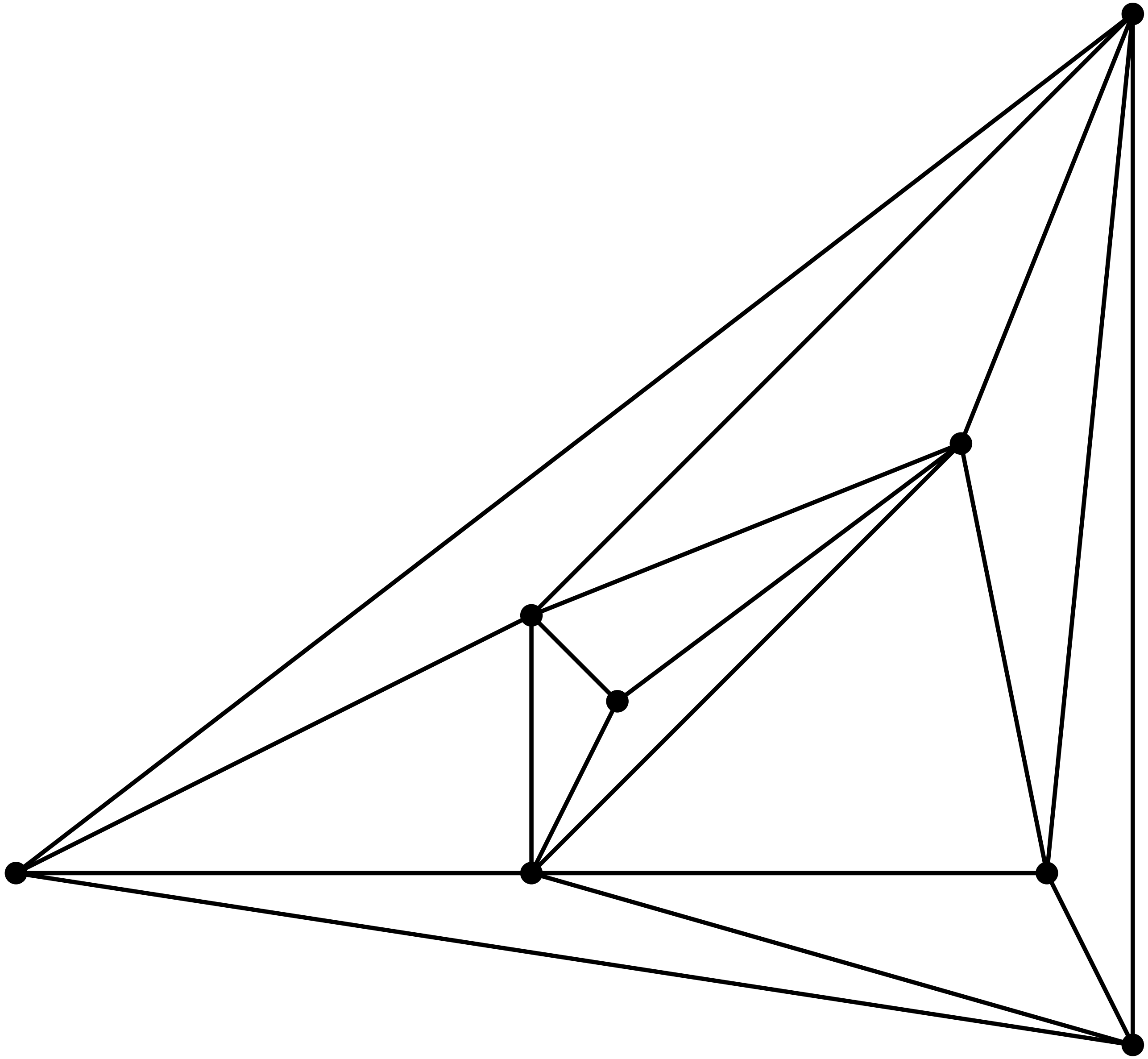
Construct exponential example



Construct exponential example



Construct exponential example



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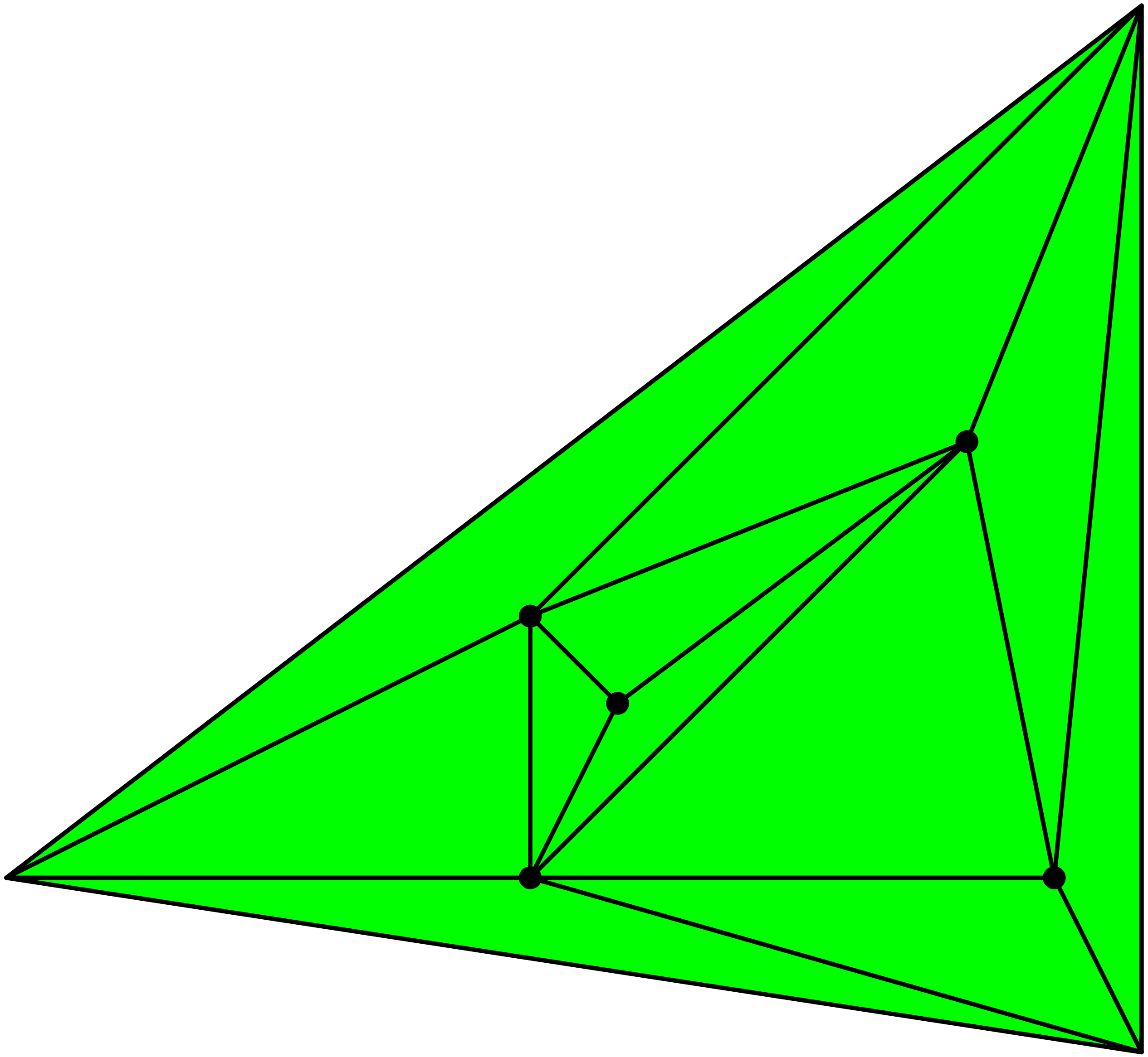

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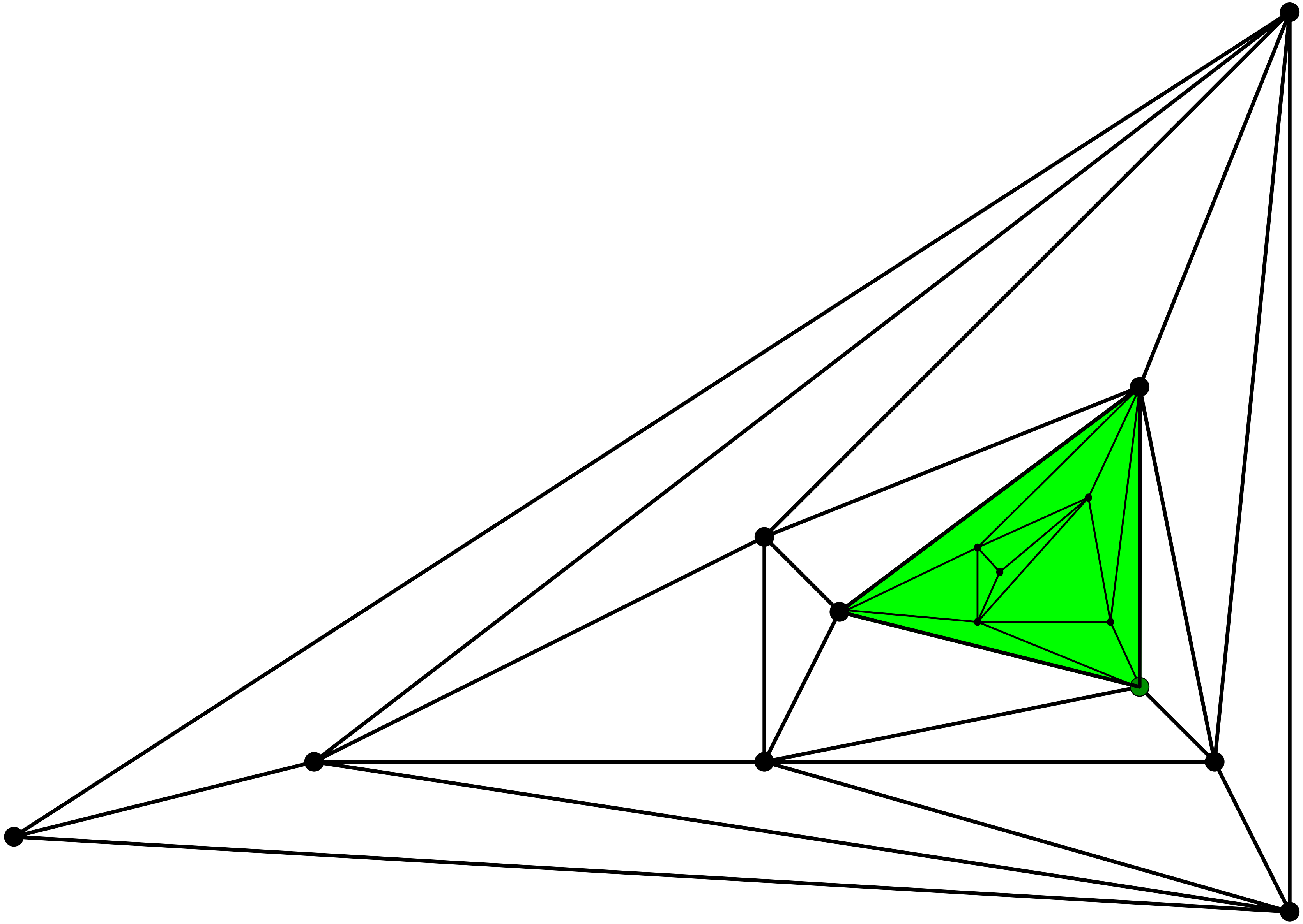
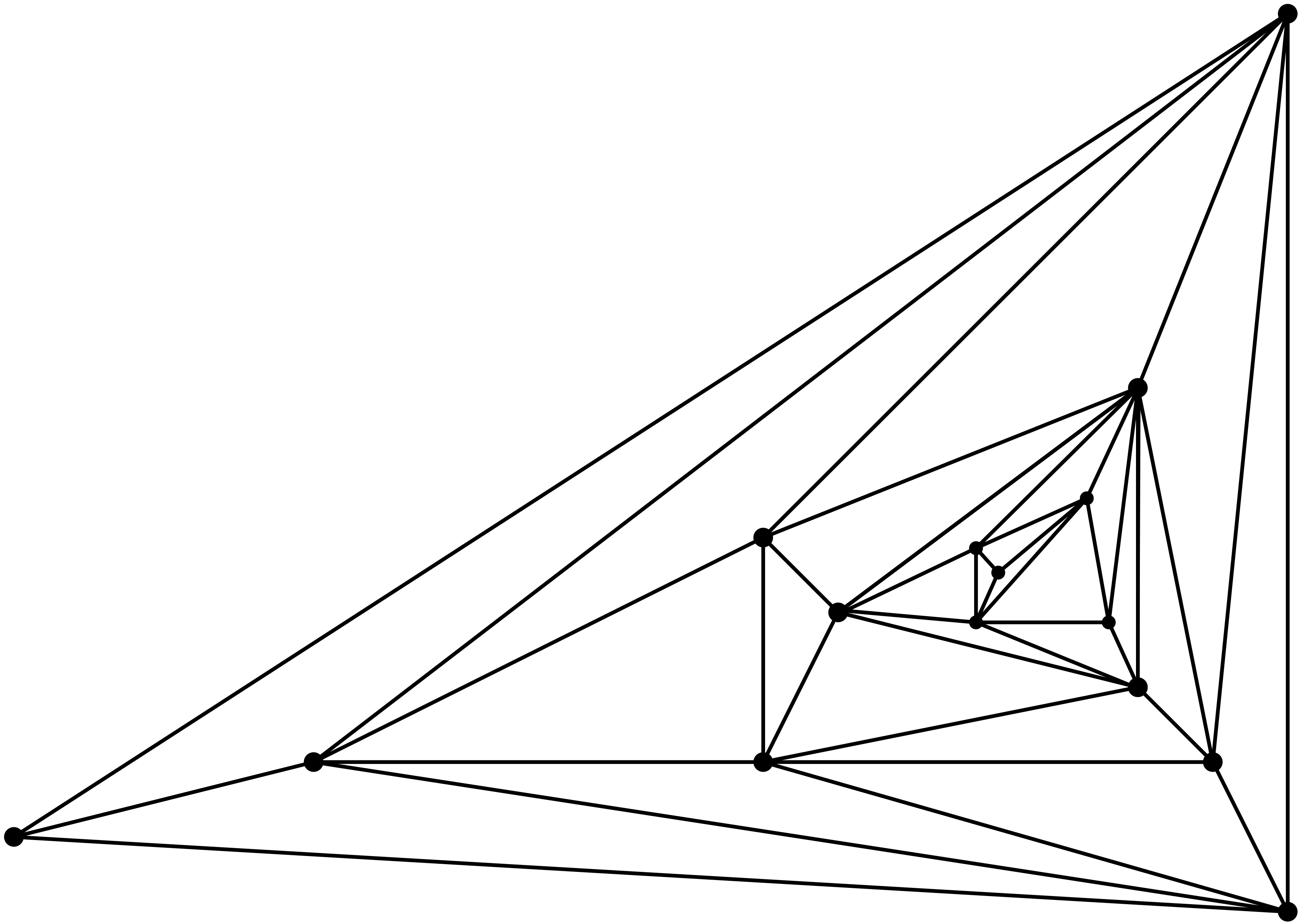
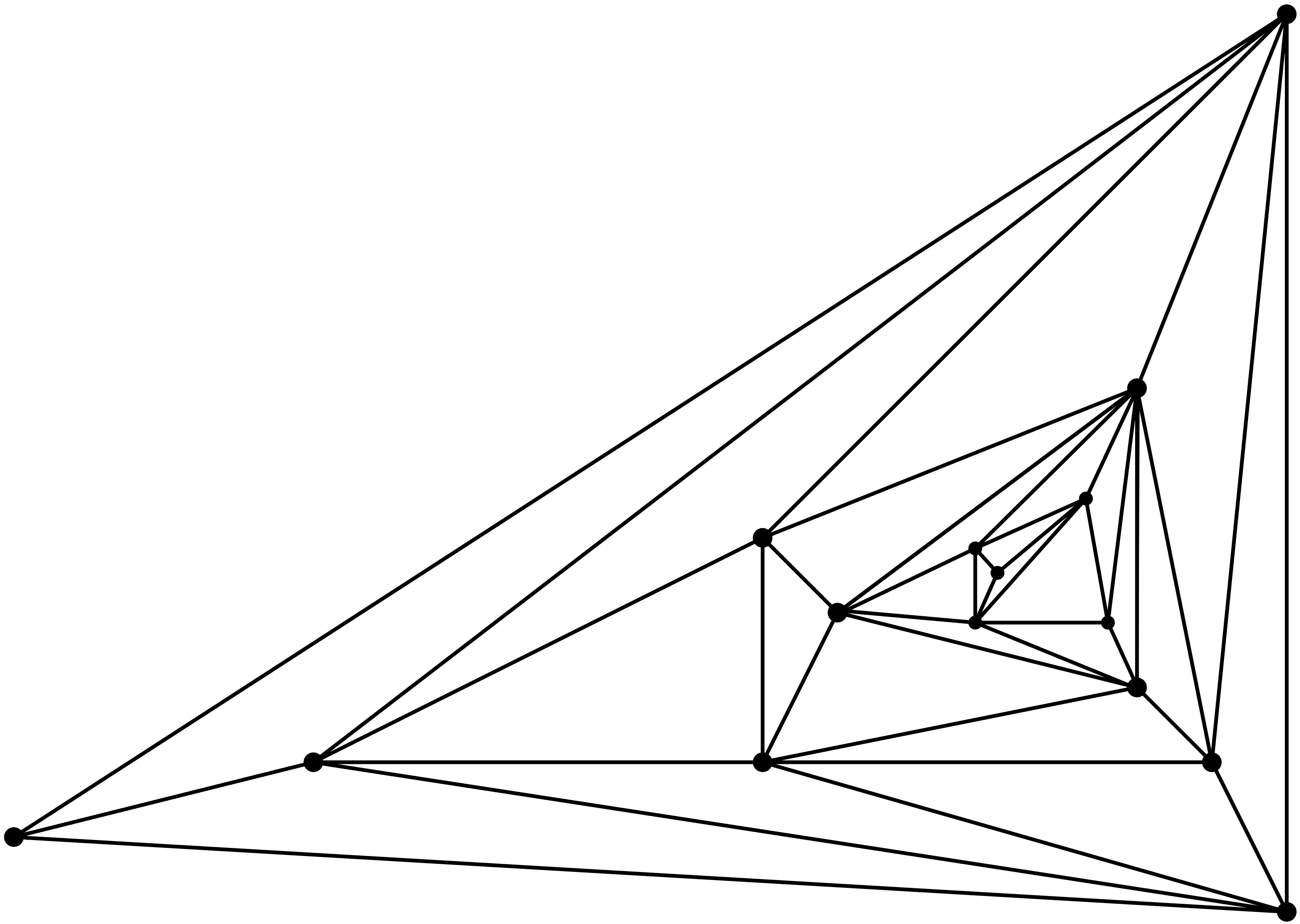


6 extra points, 12 extra rooms



path length more than doubles!



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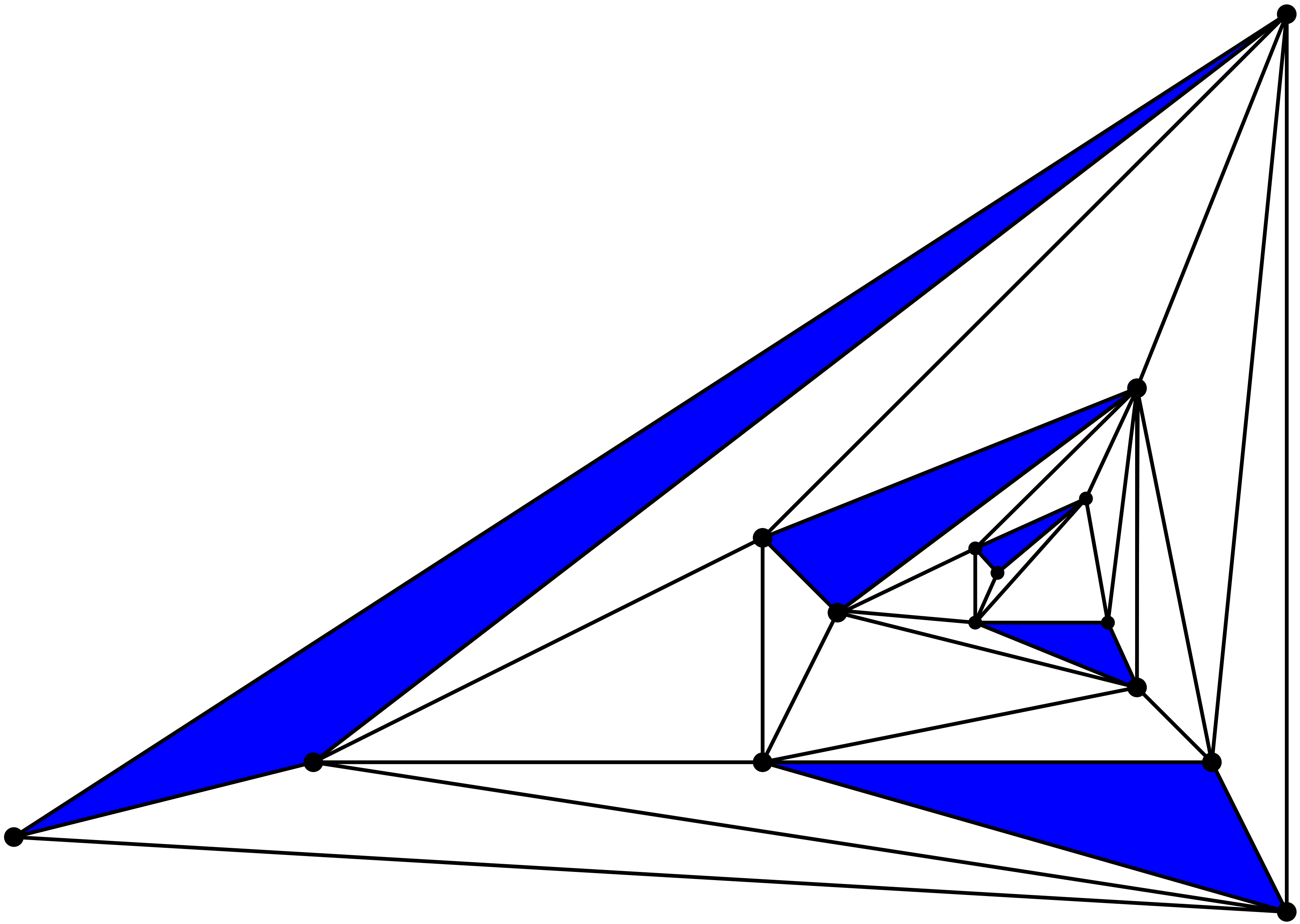

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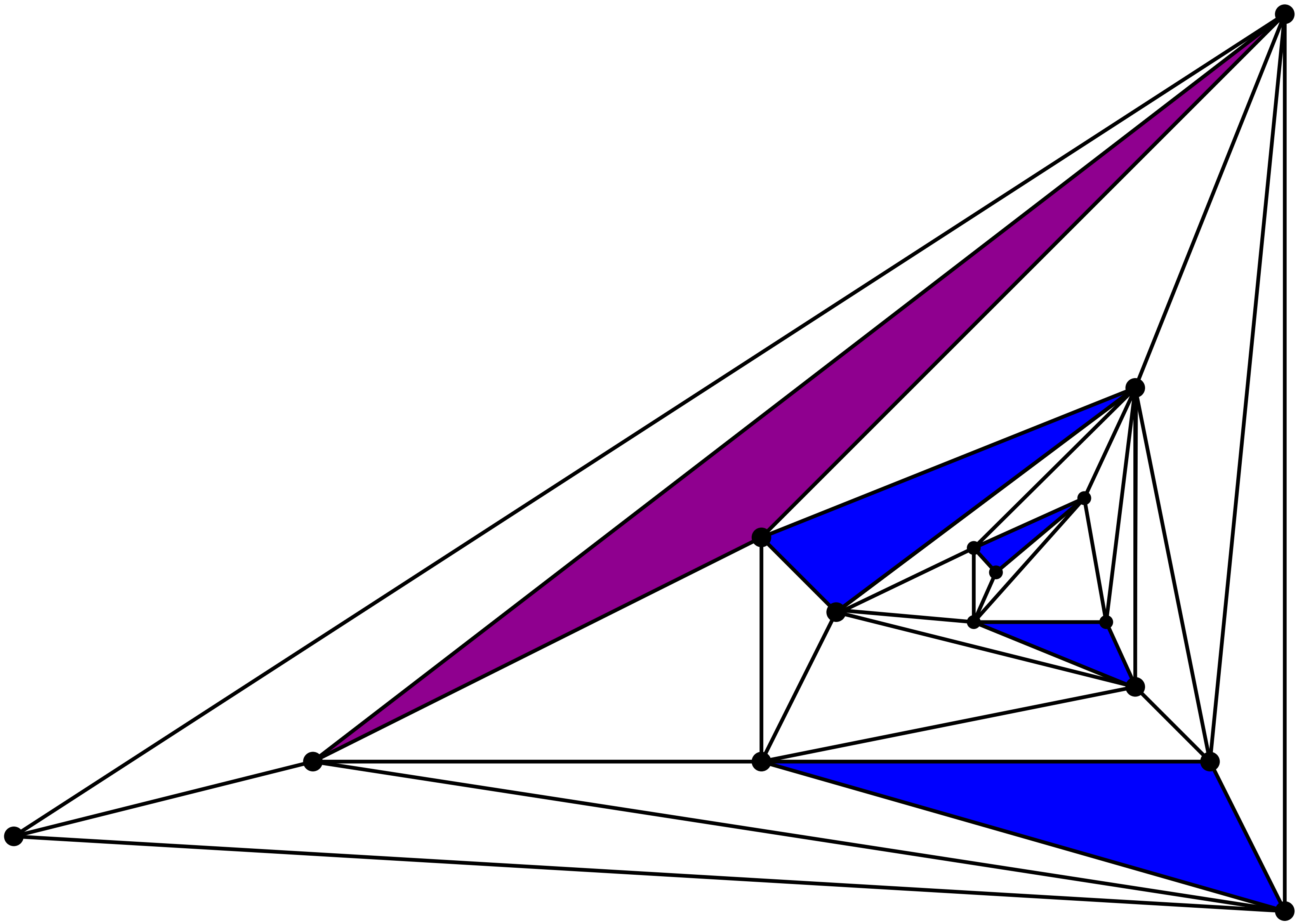

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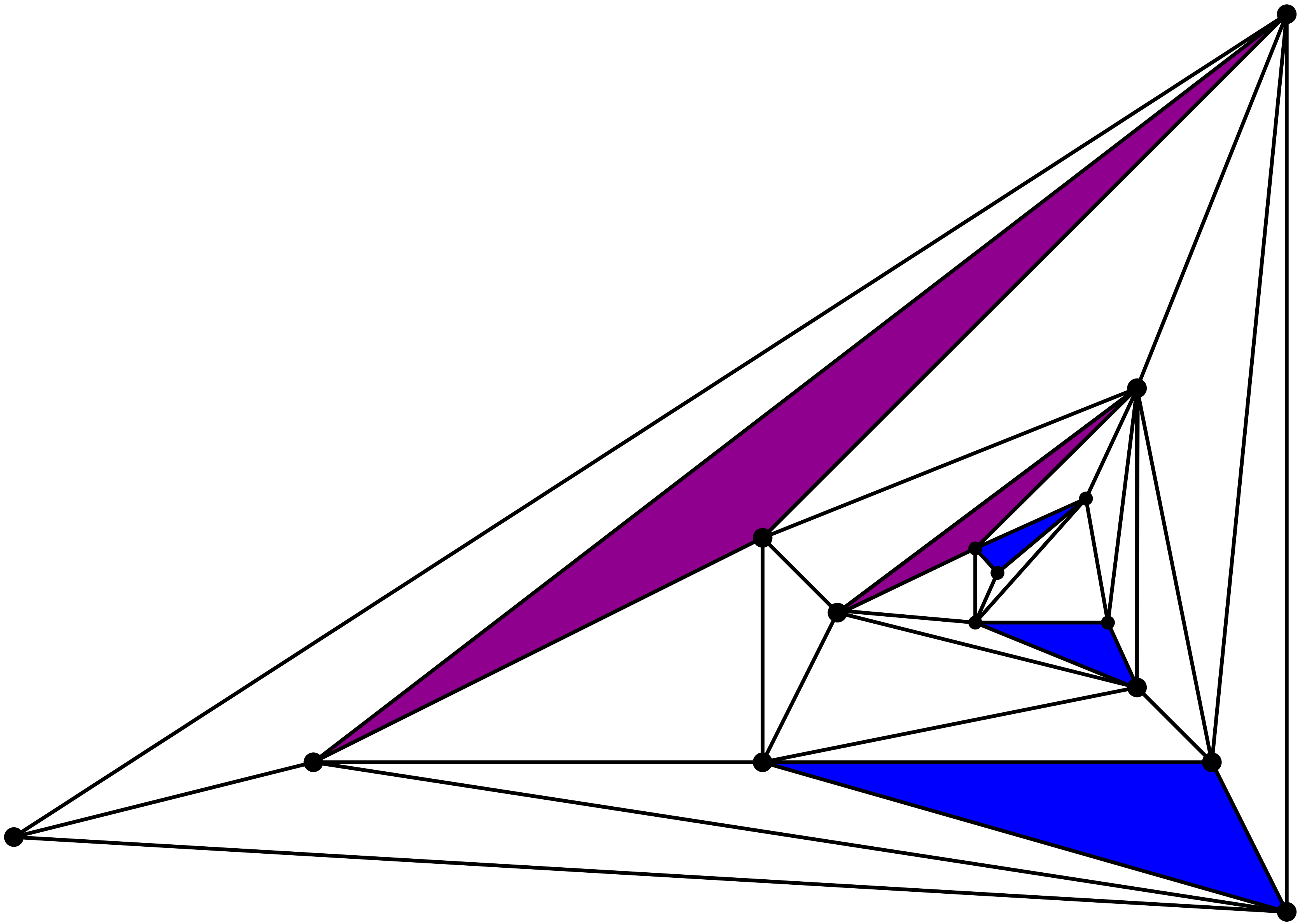

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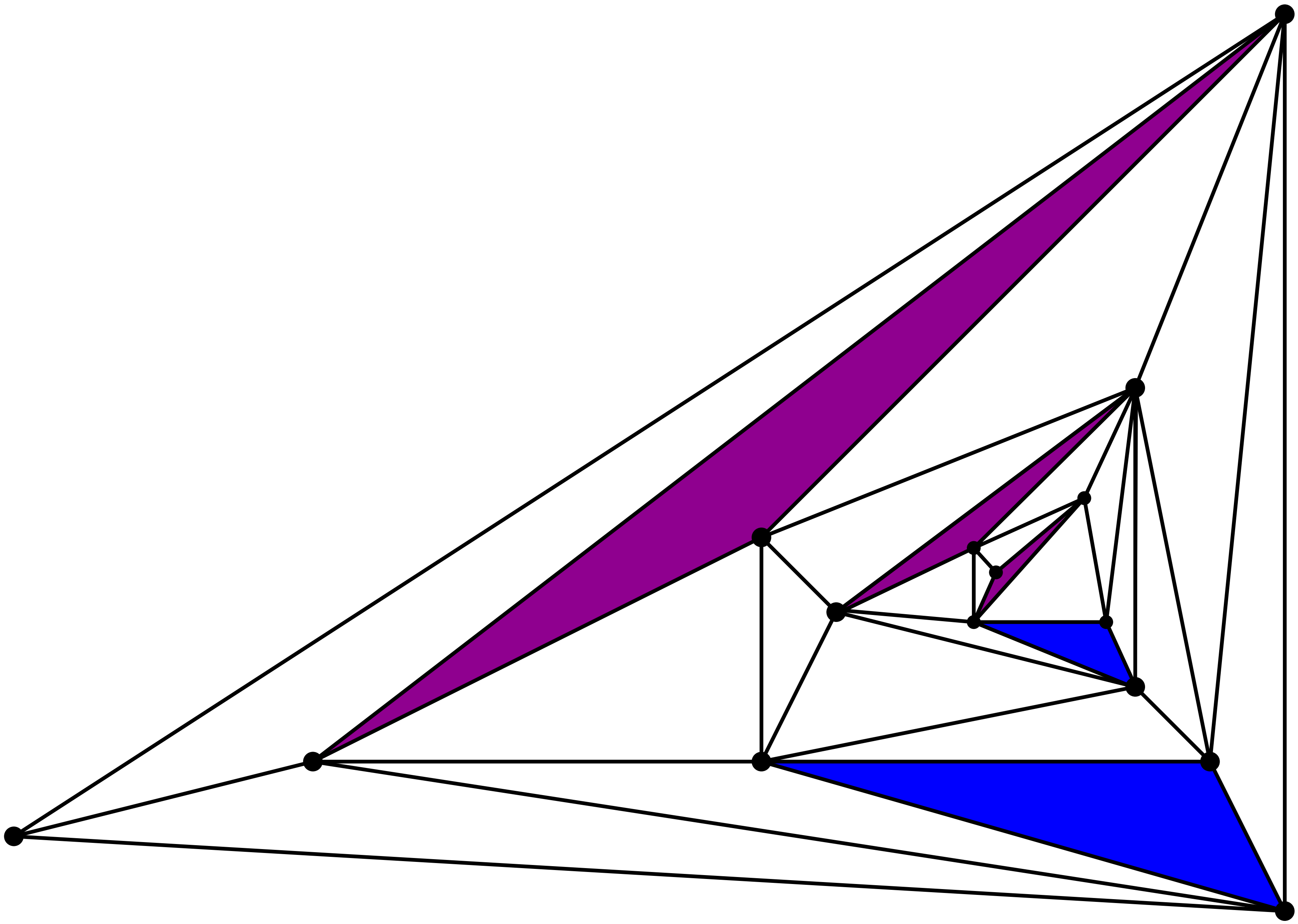

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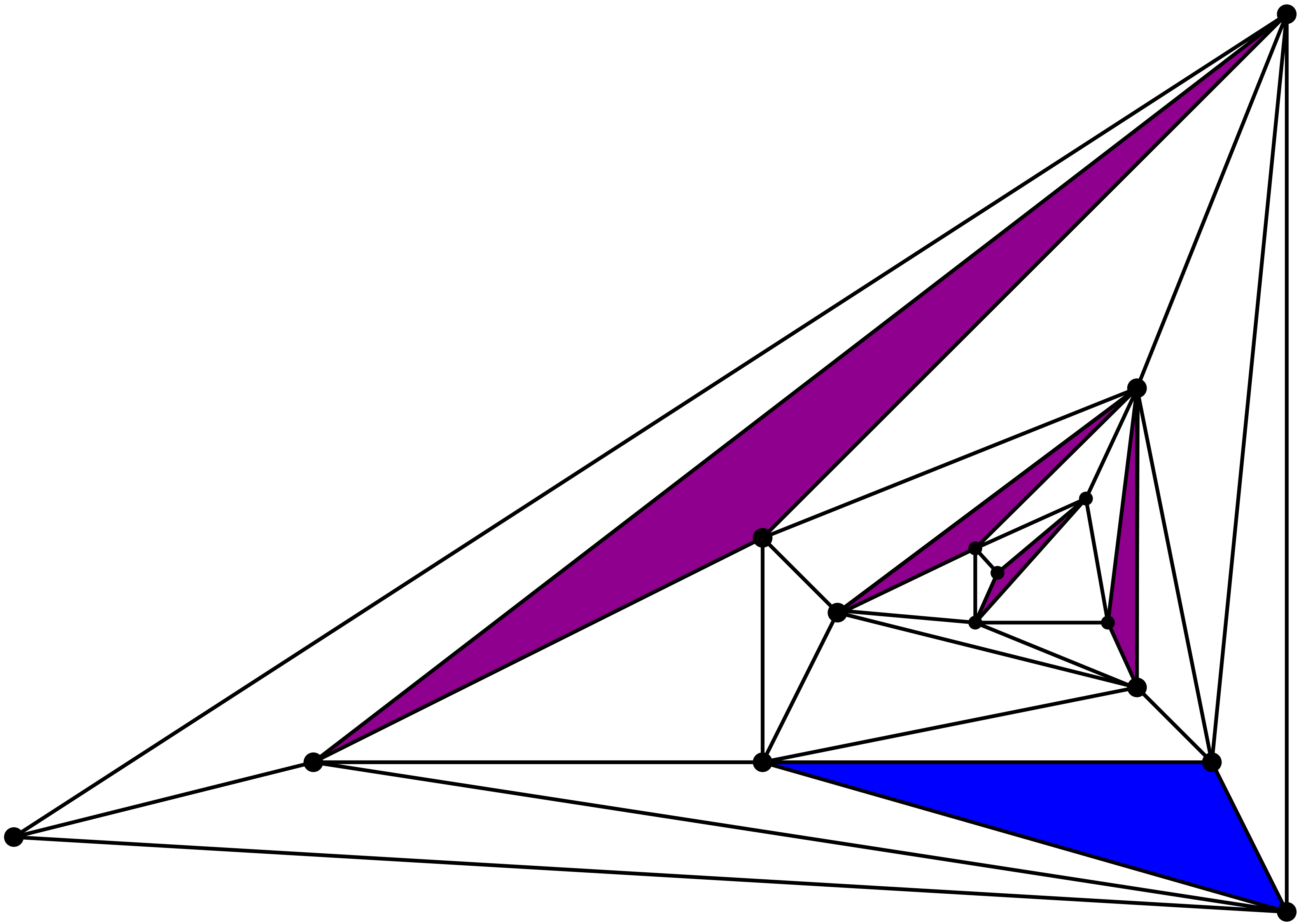

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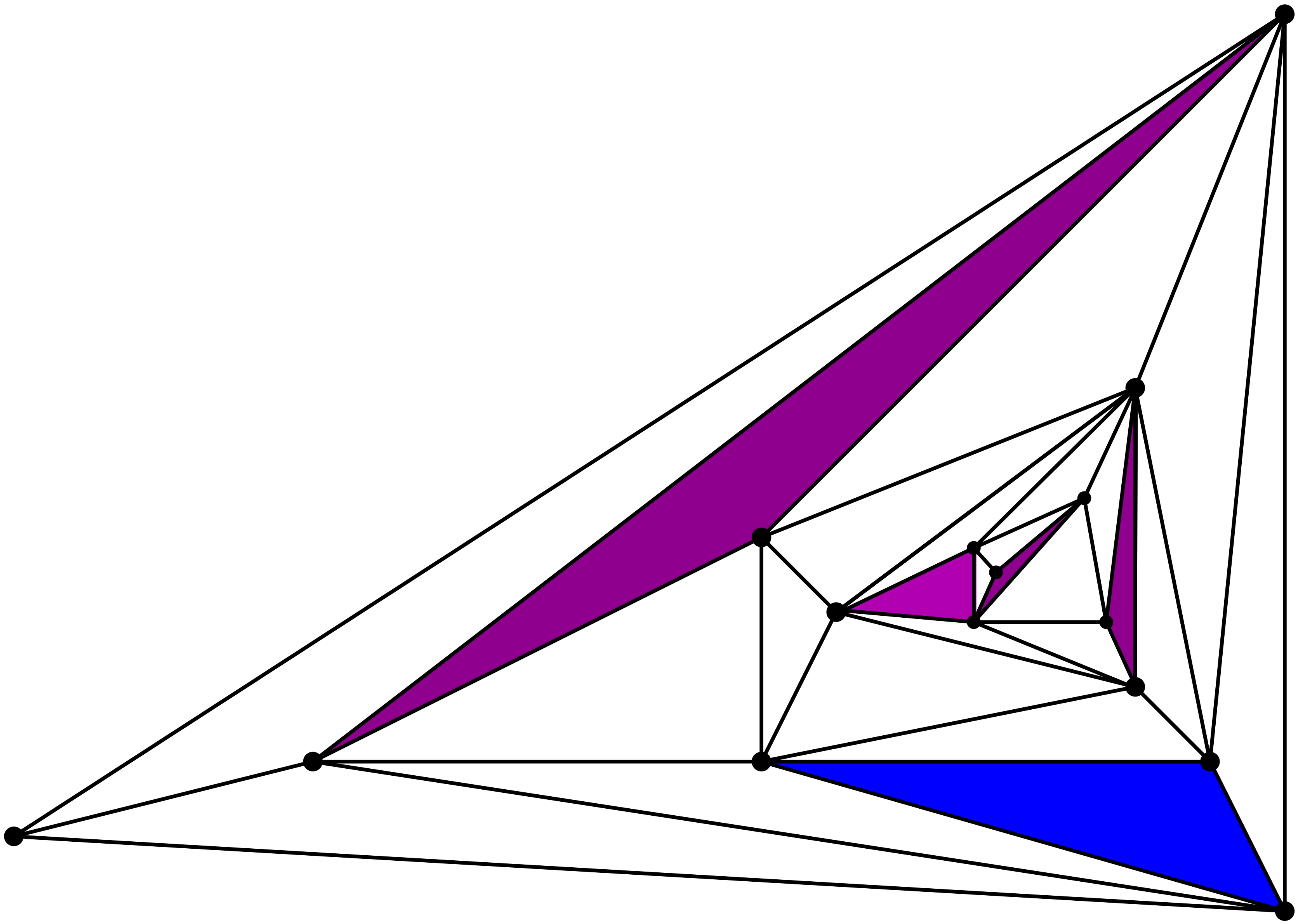

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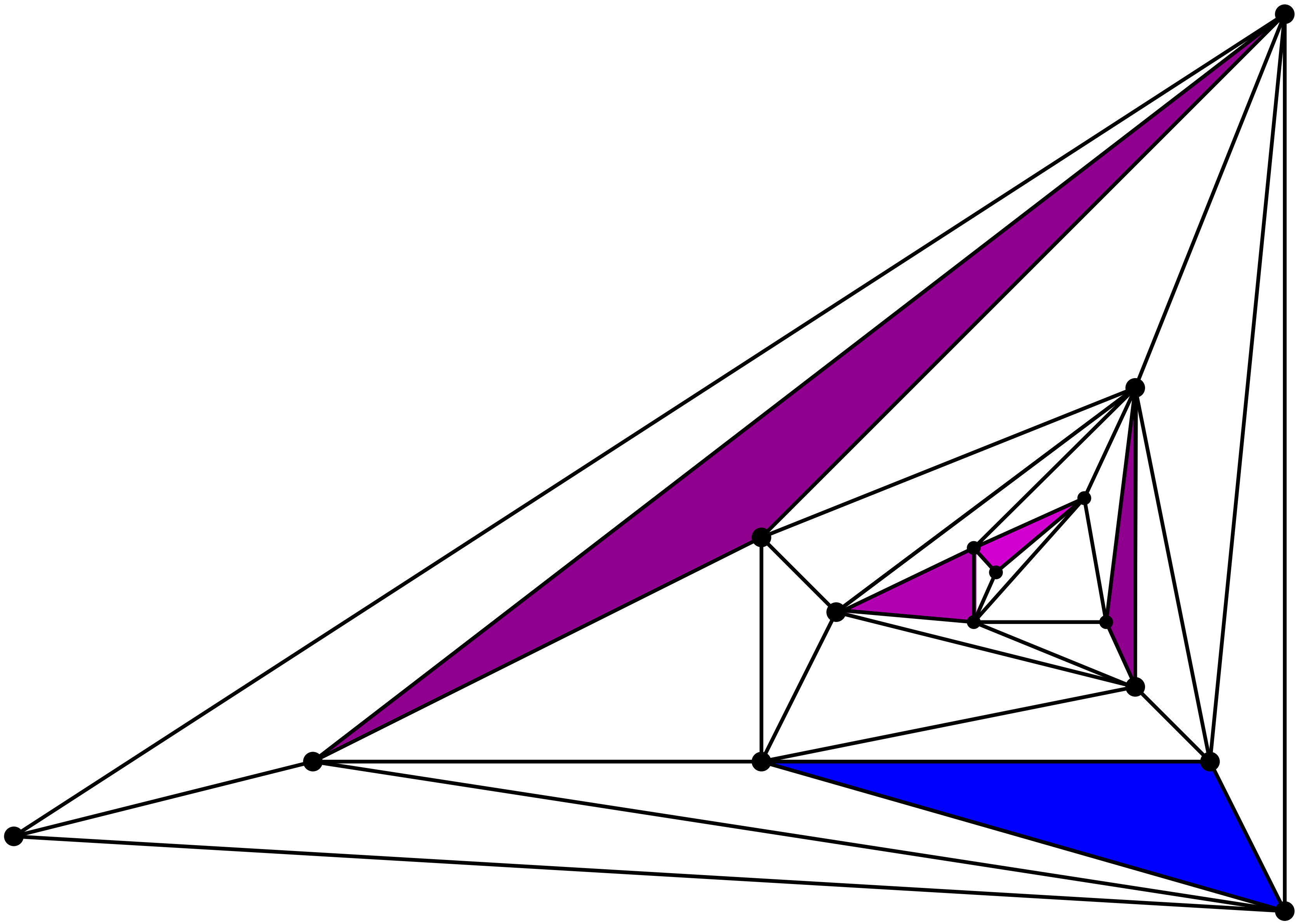

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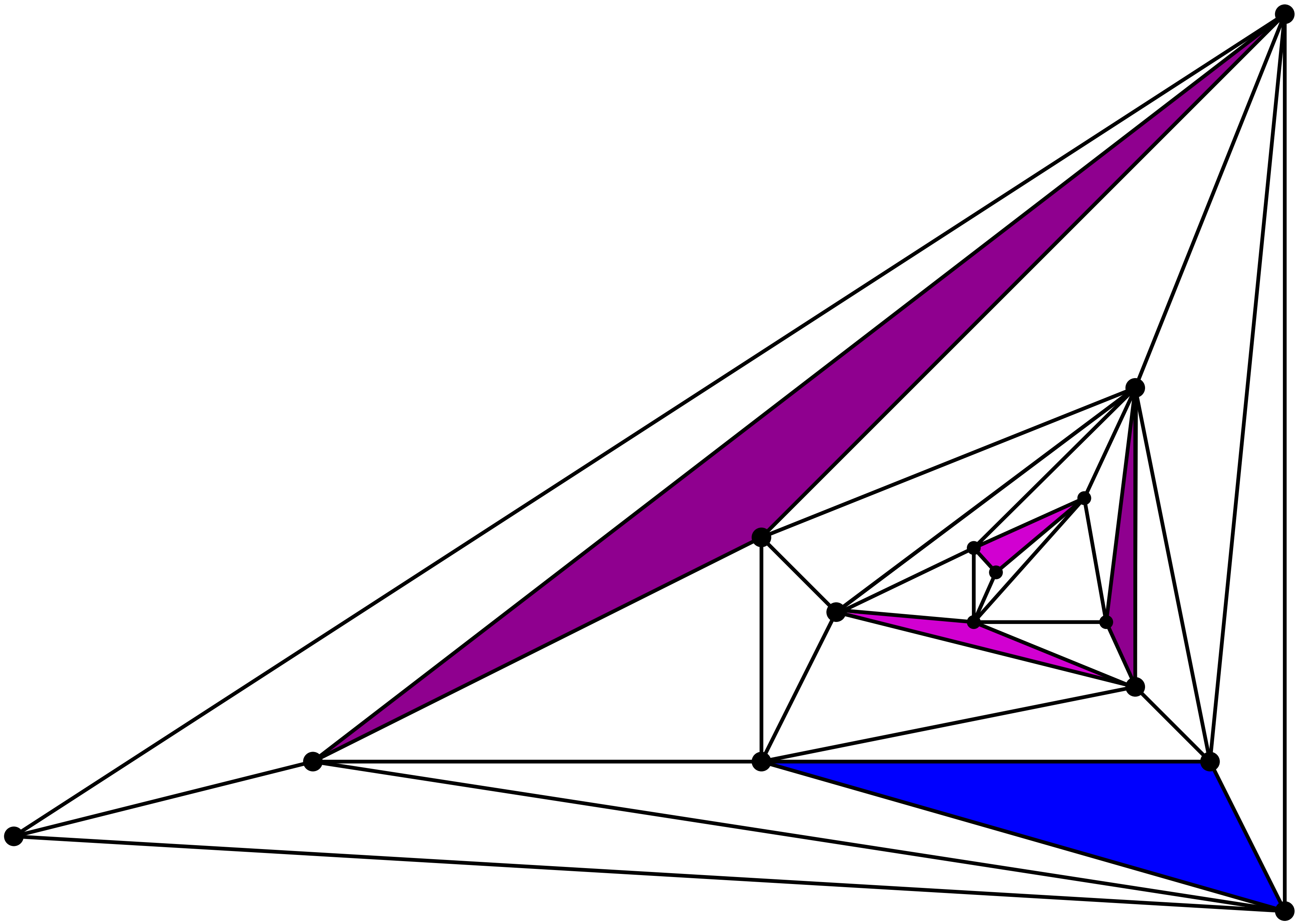

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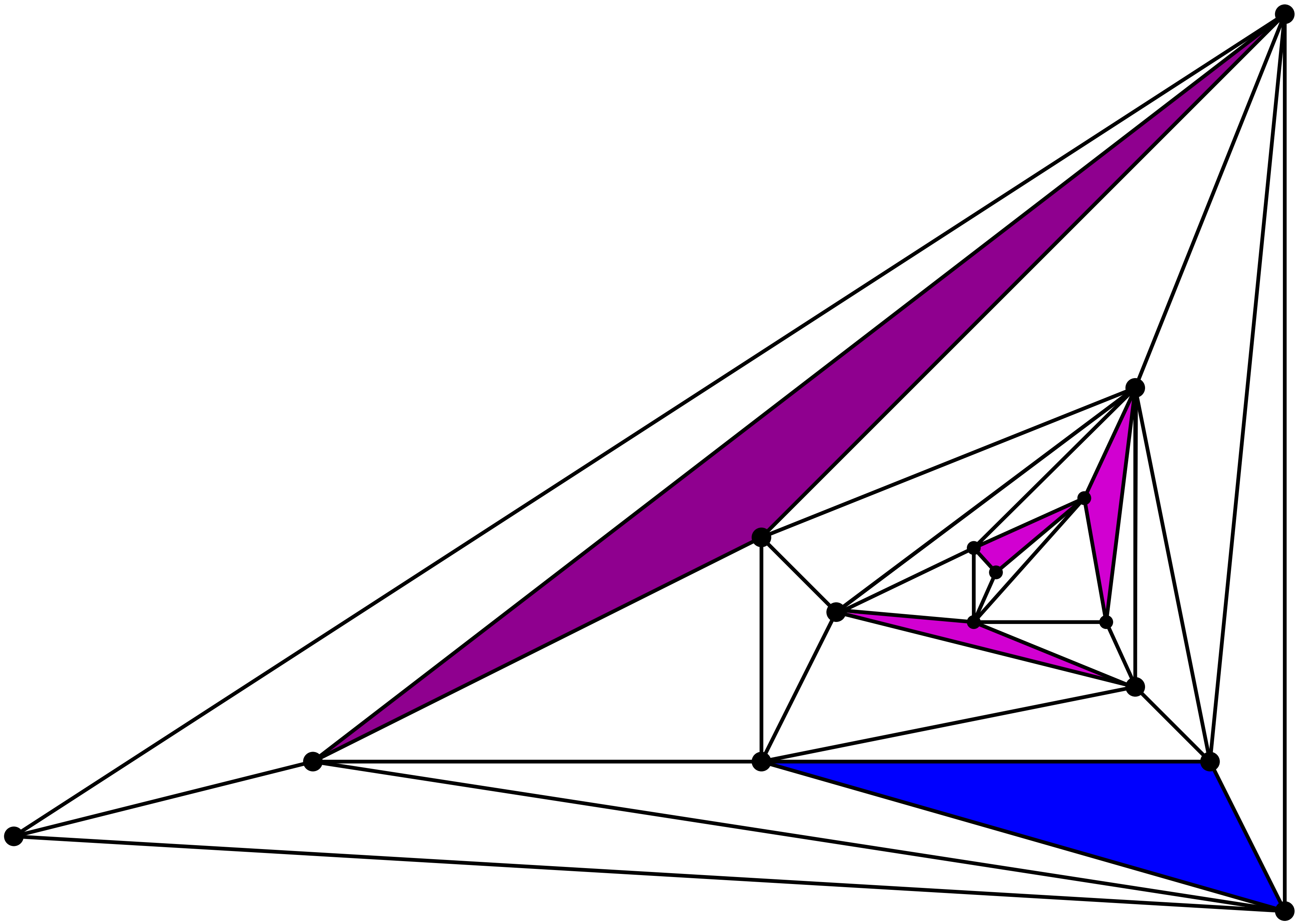

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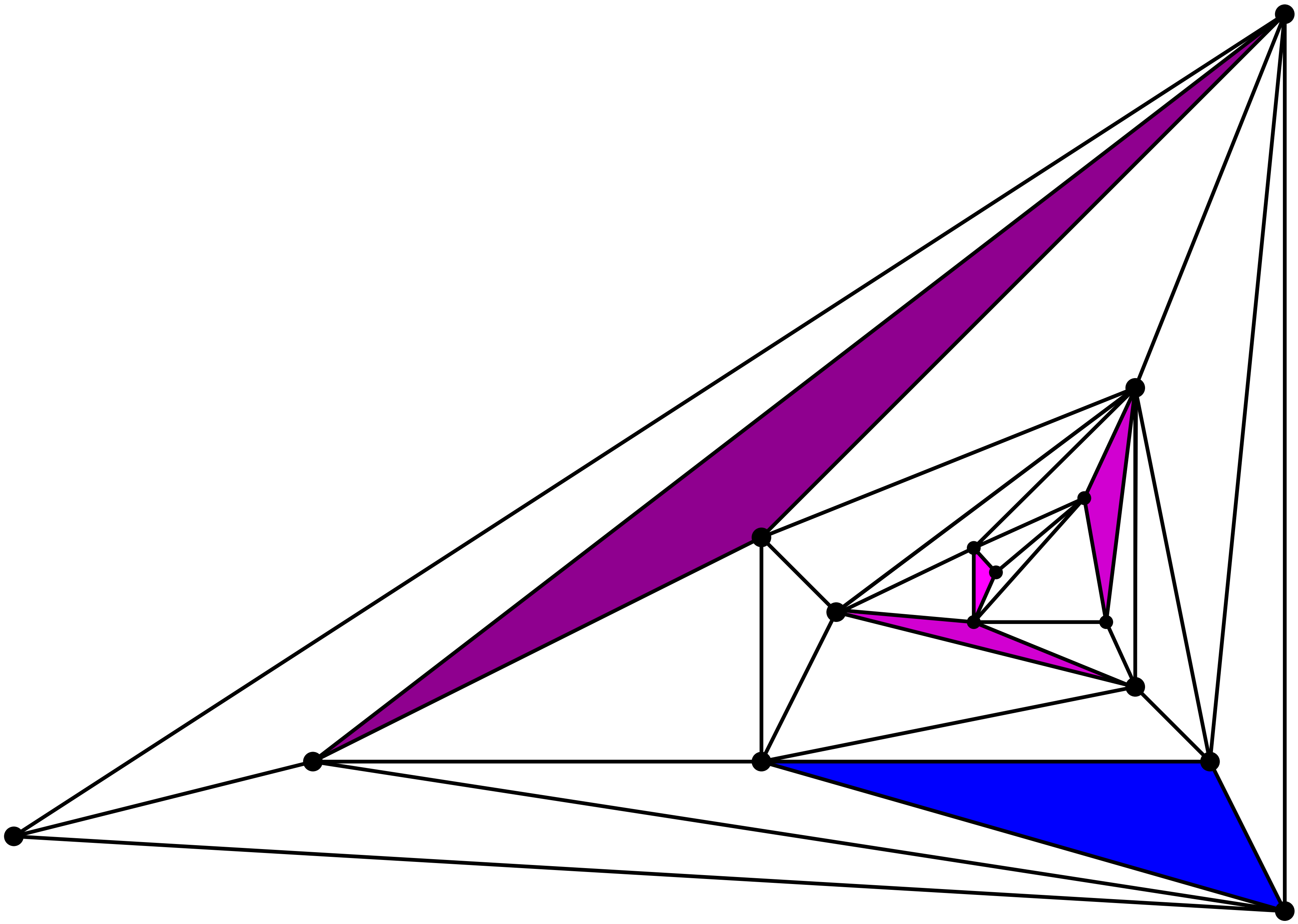

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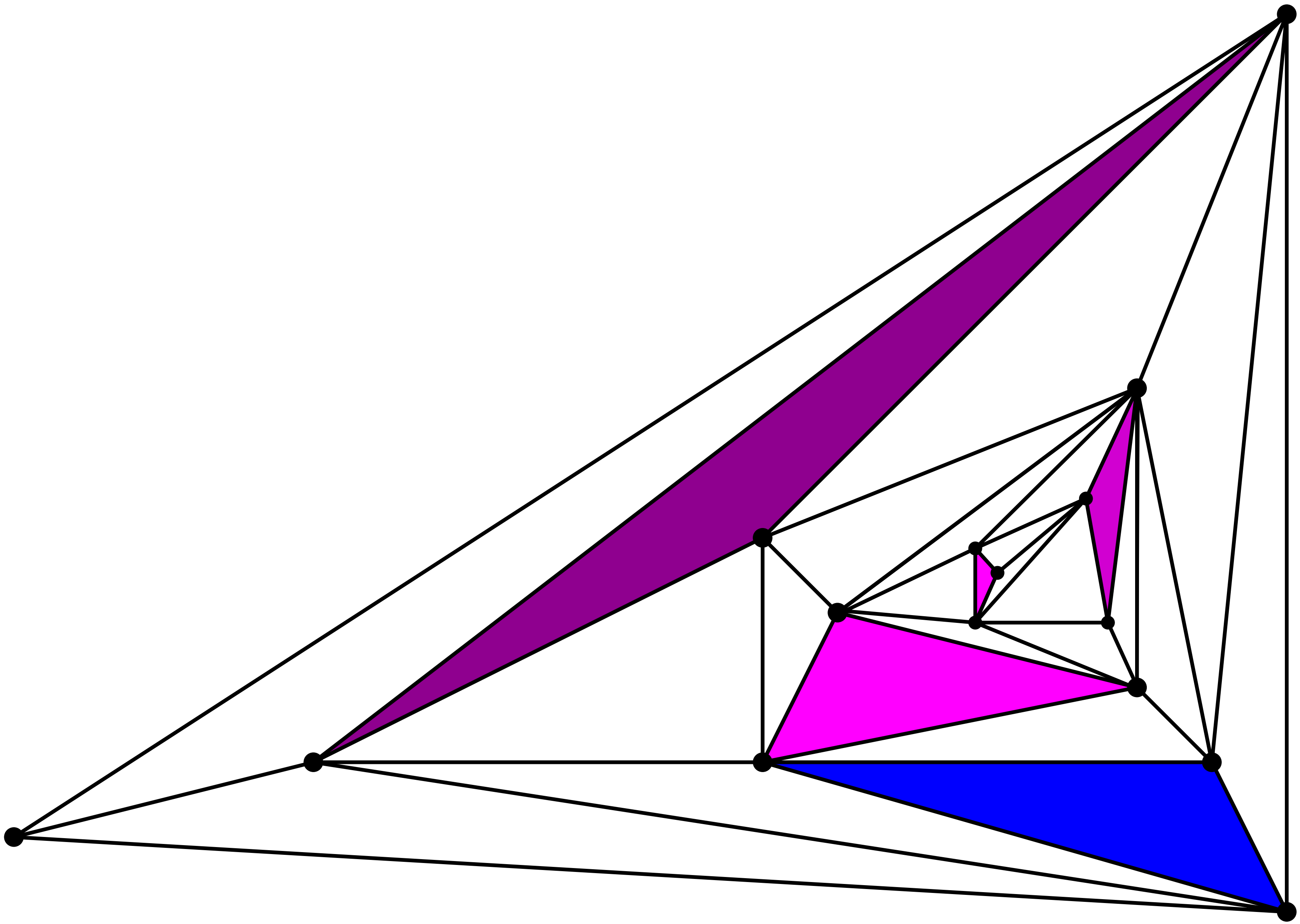

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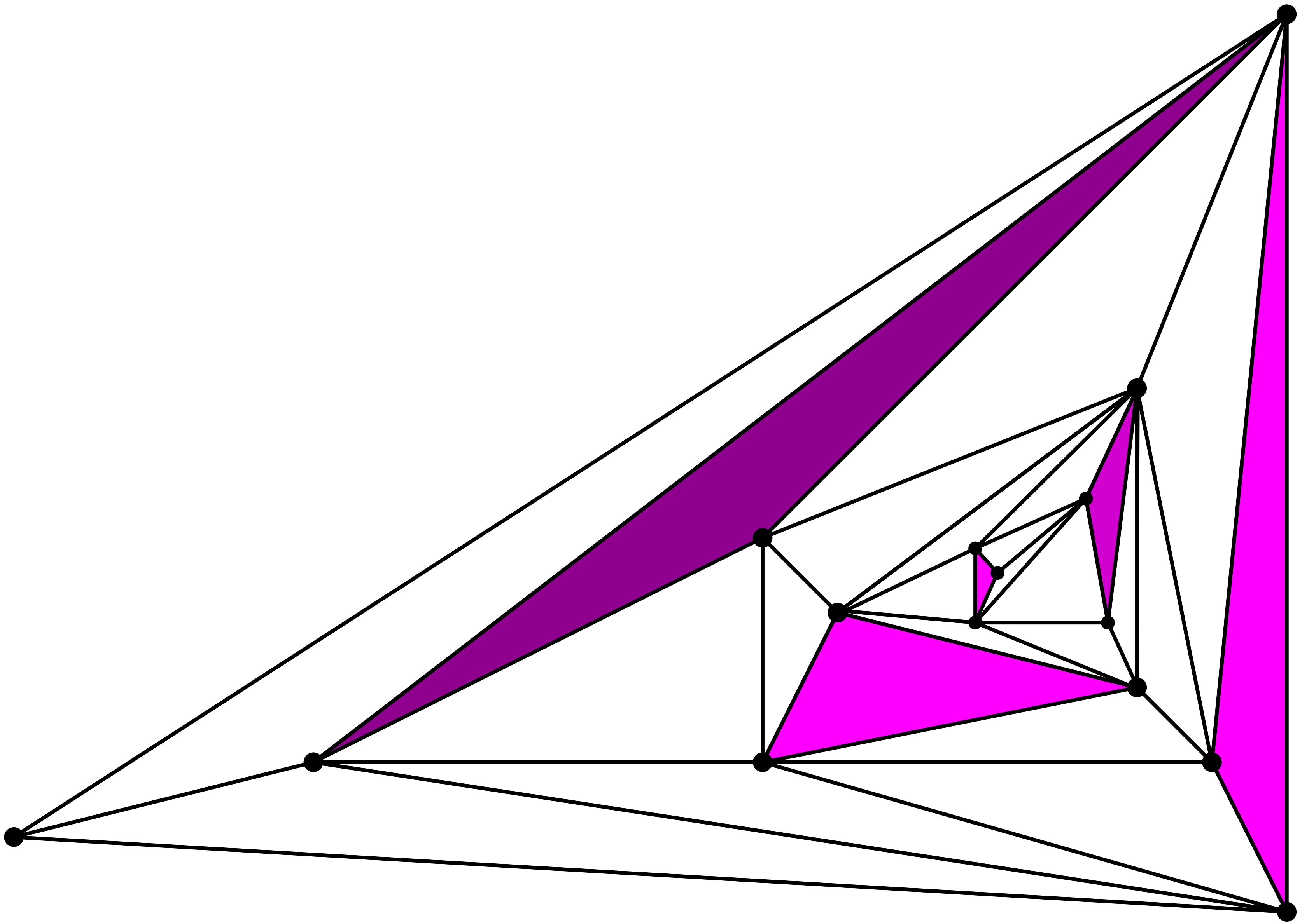

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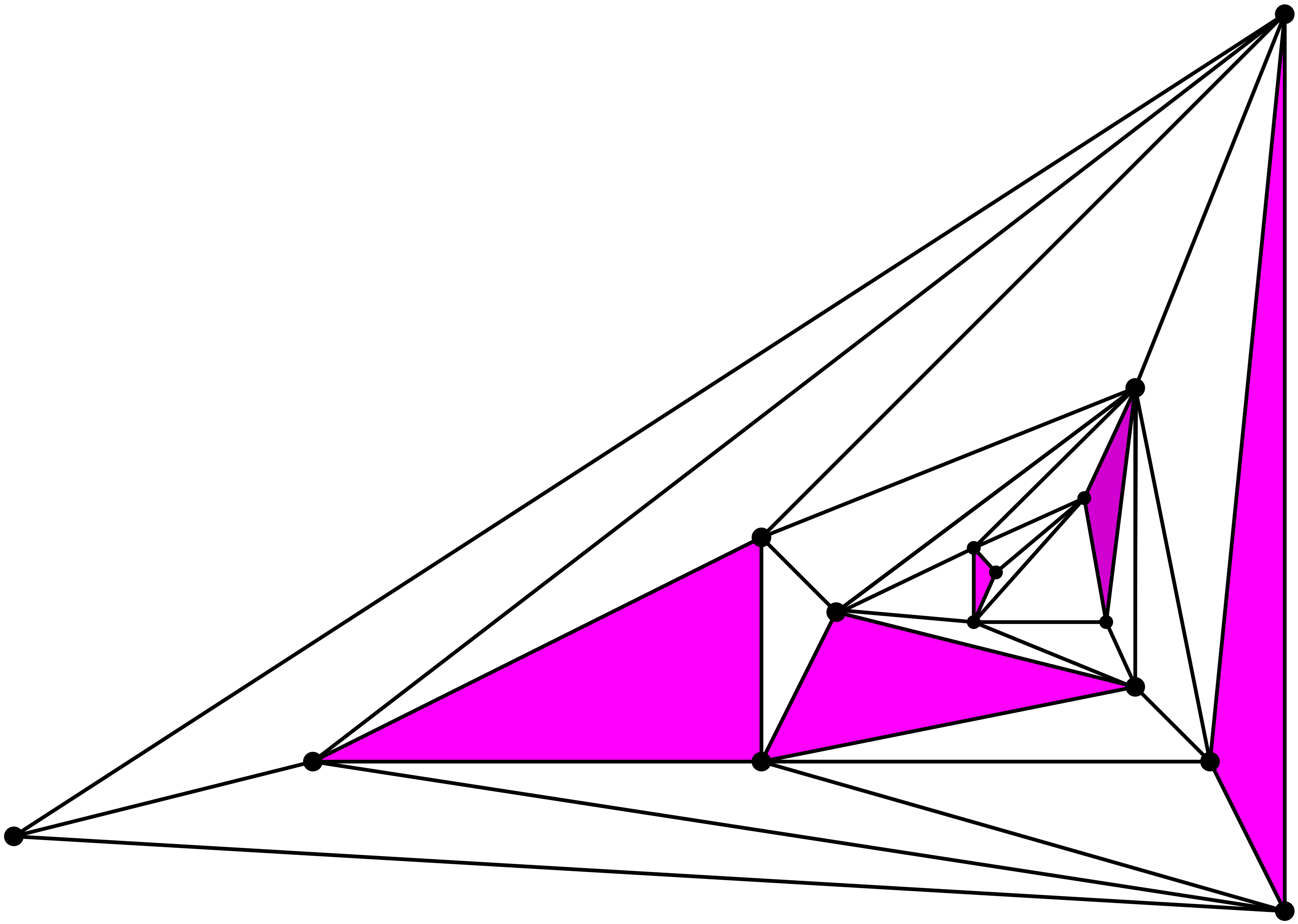

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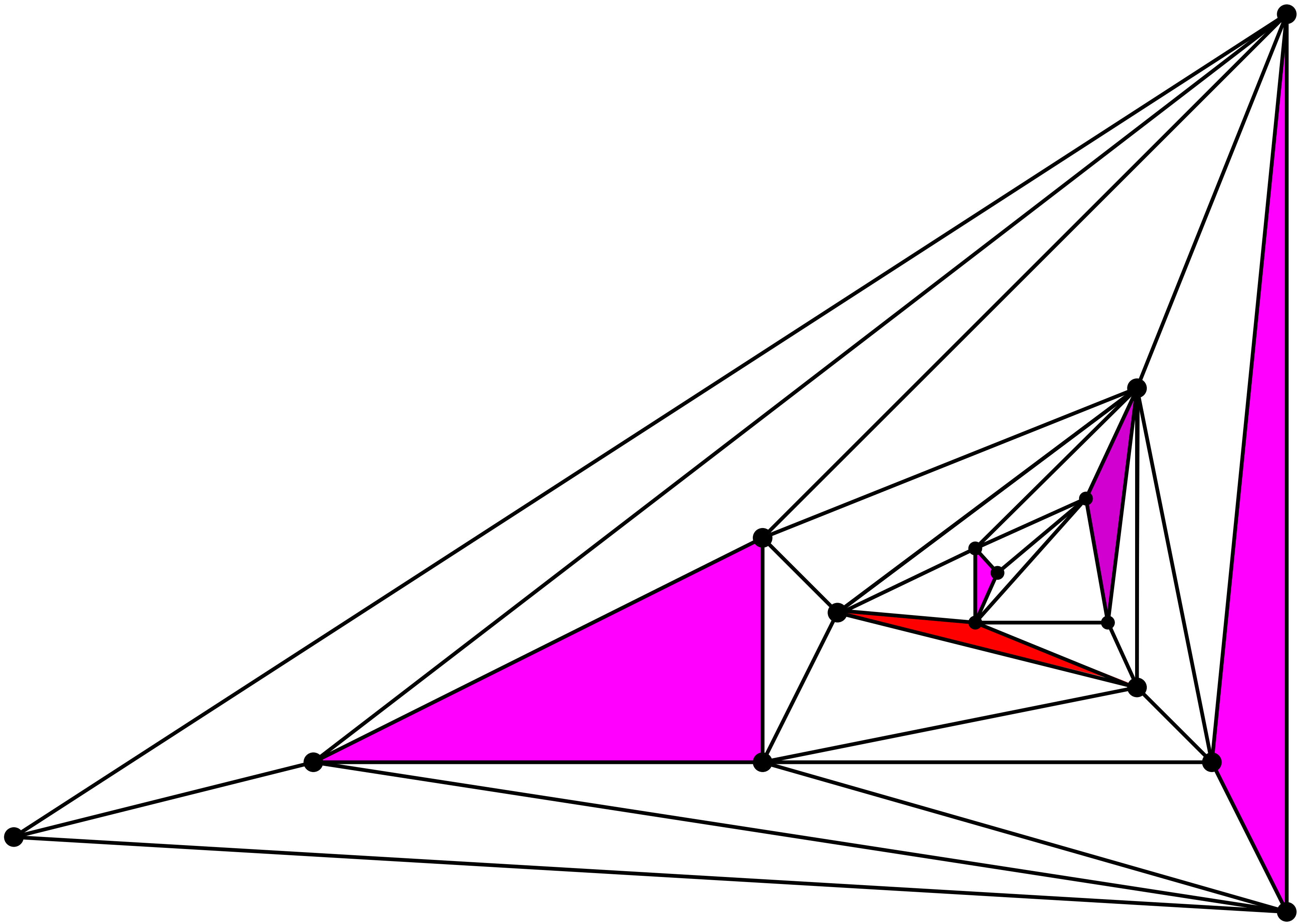

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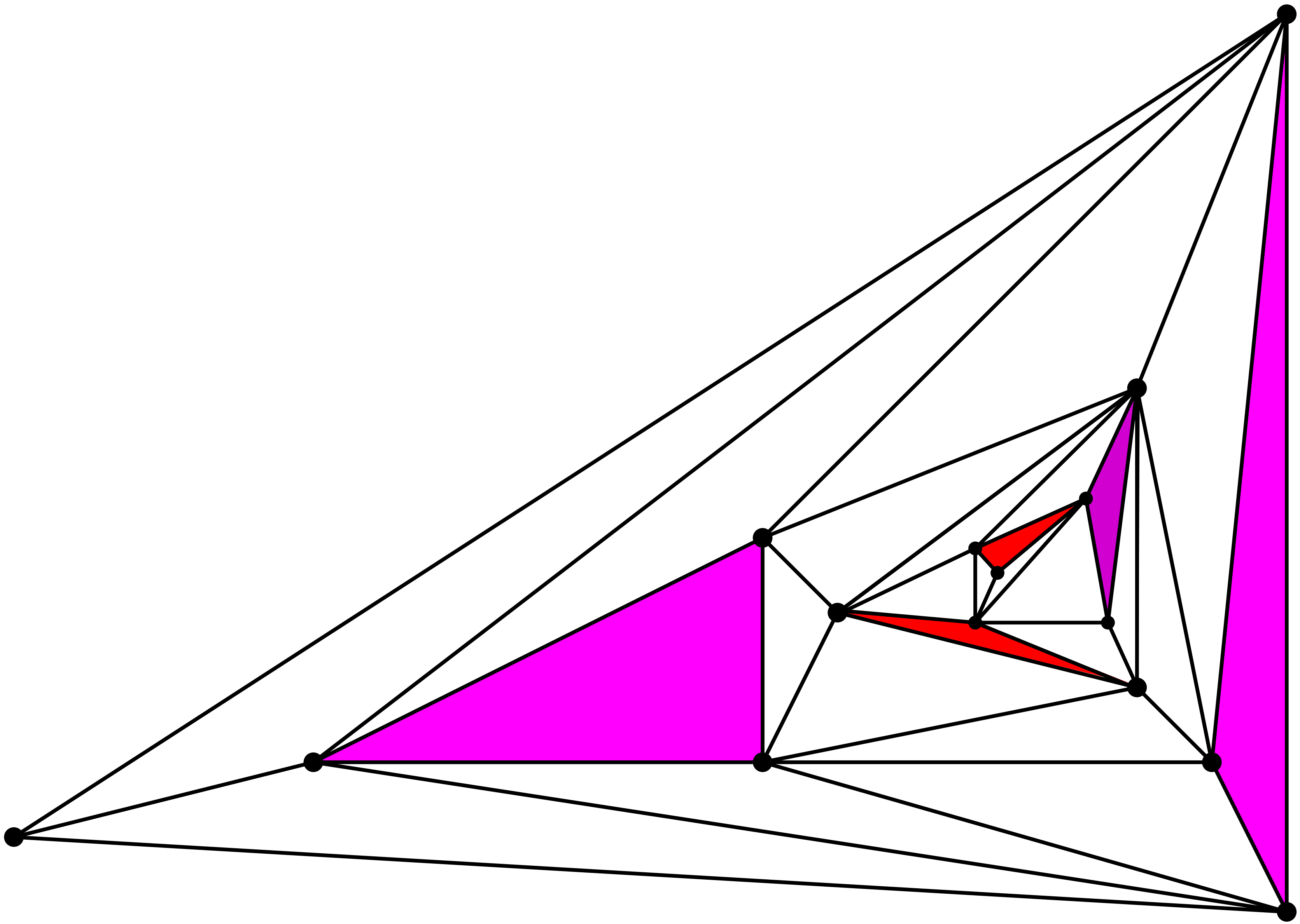

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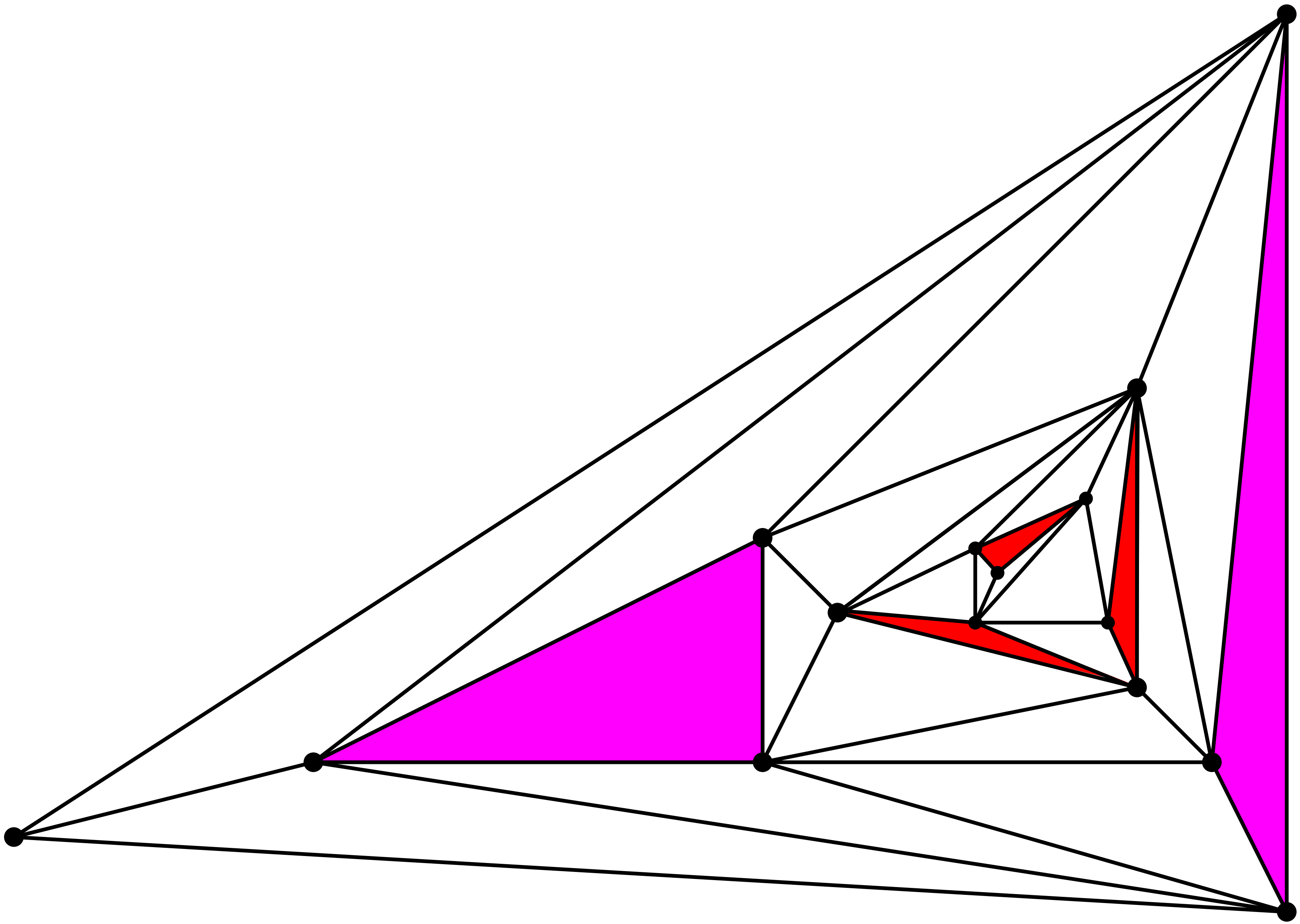

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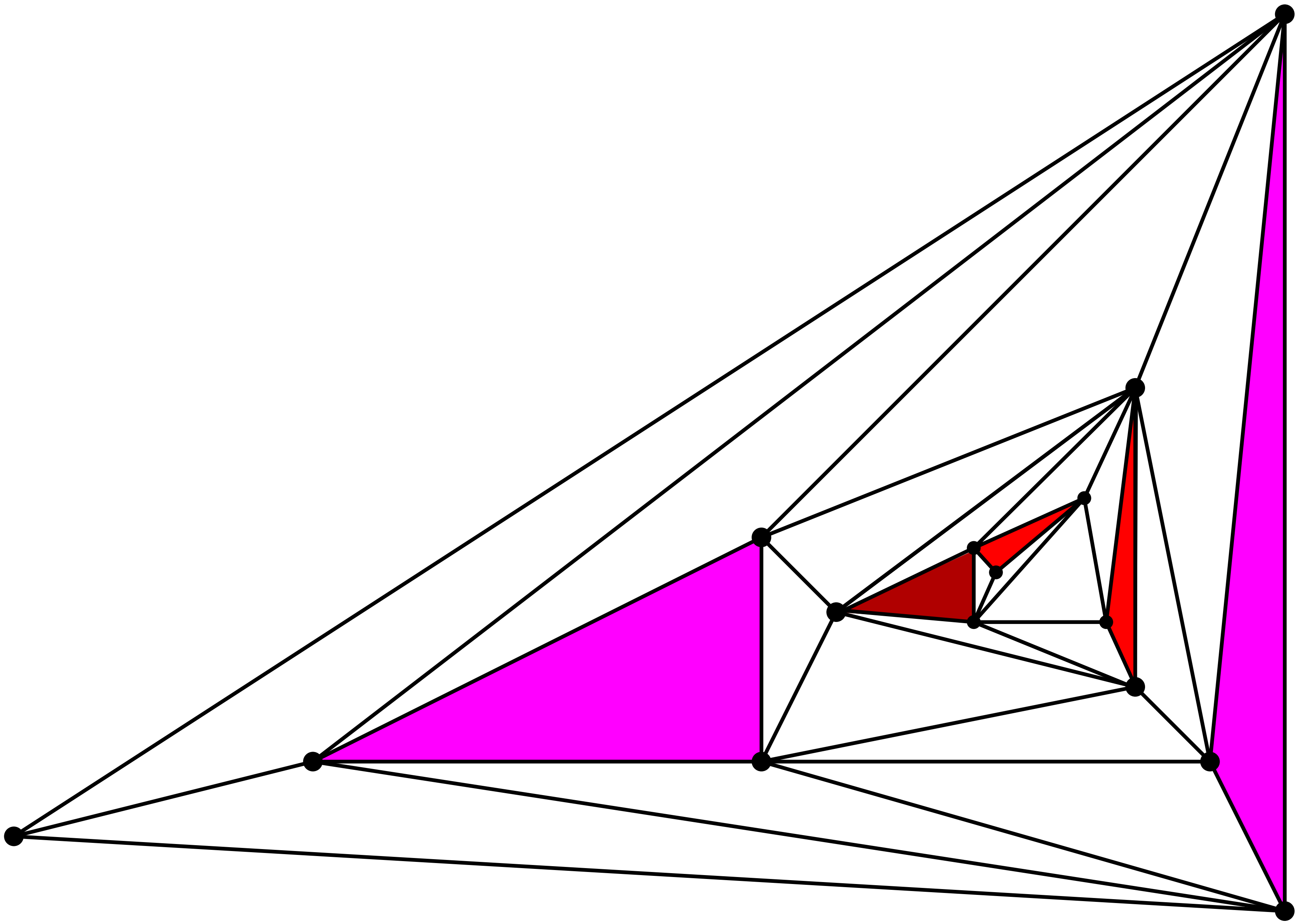

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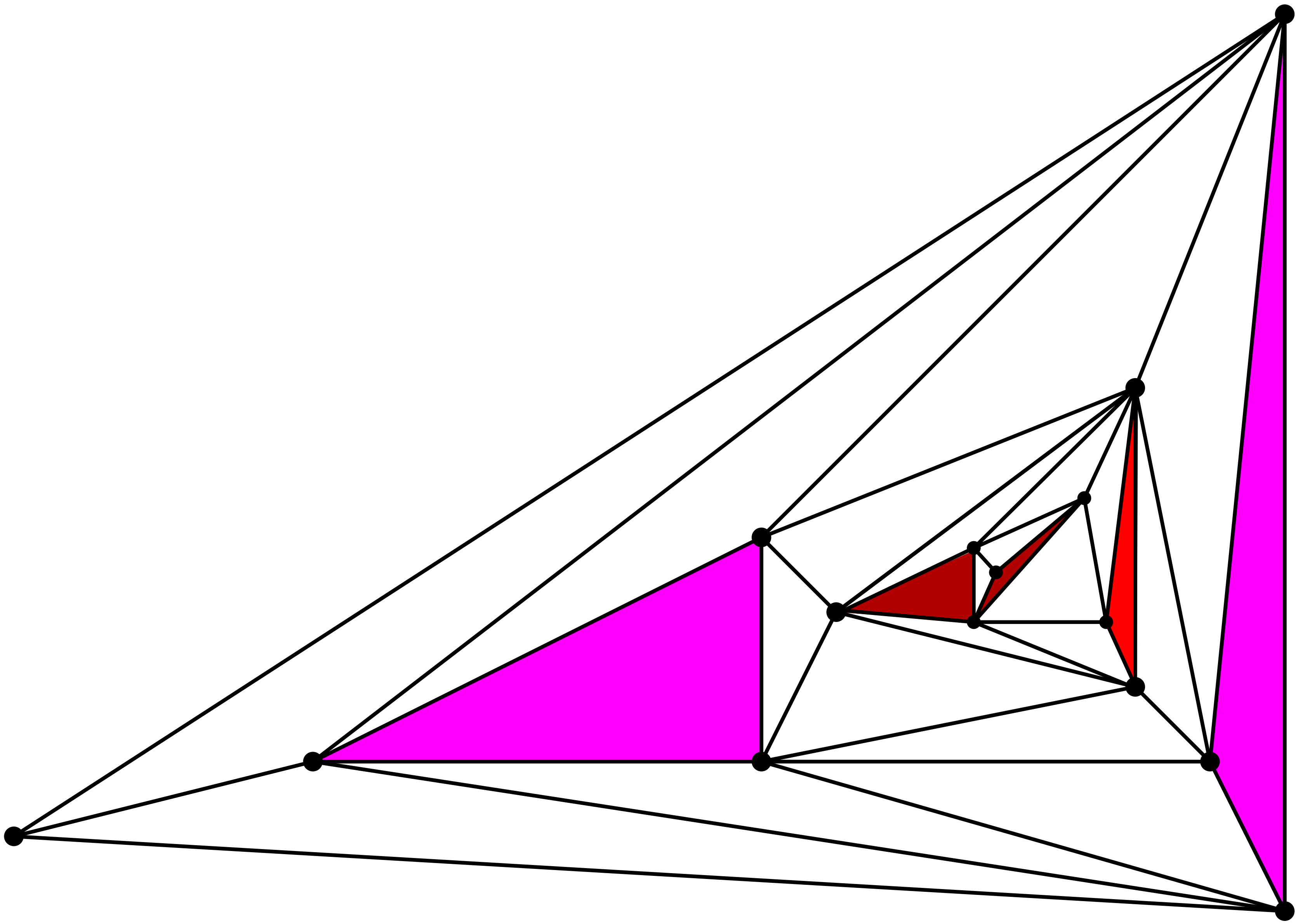

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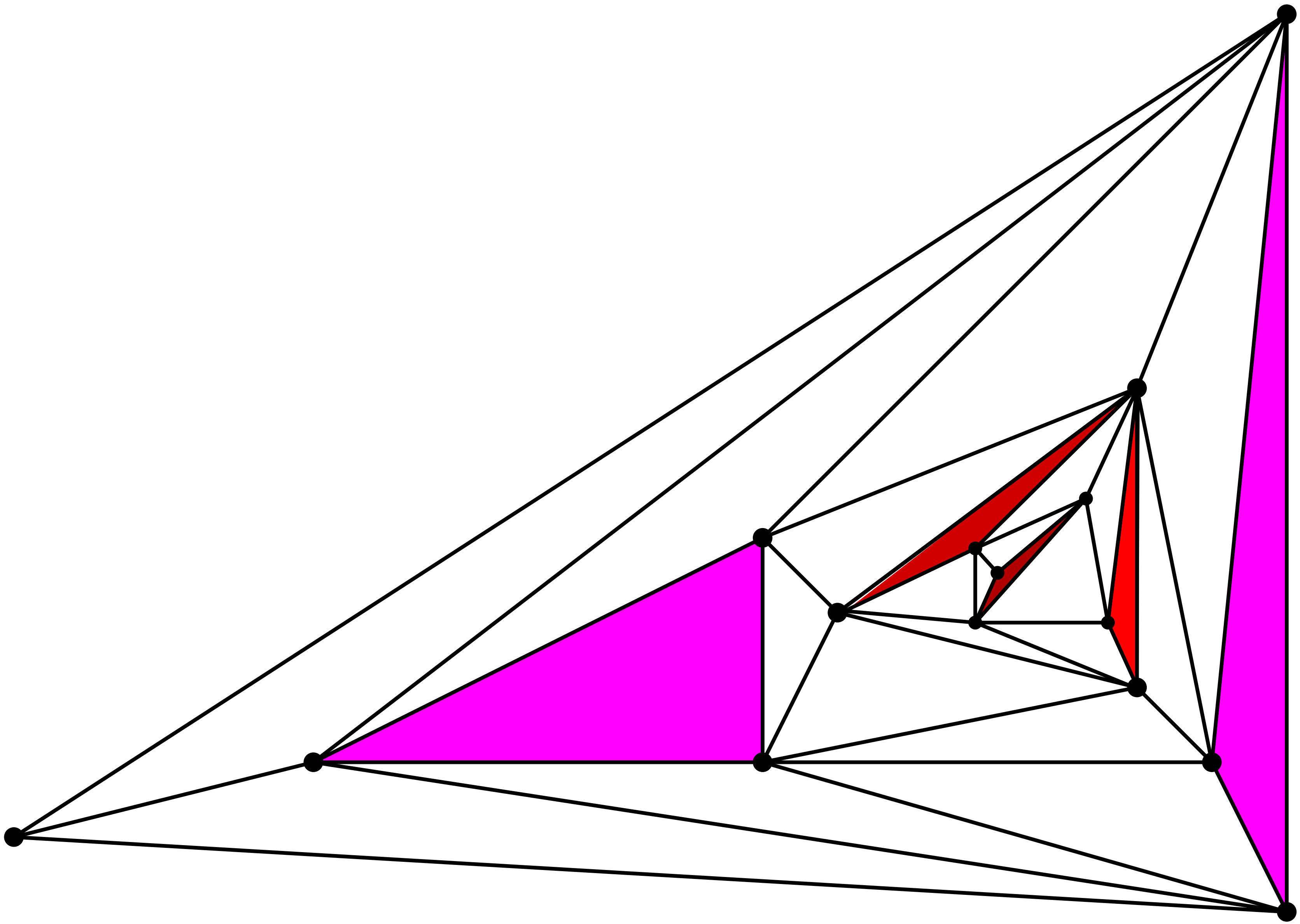

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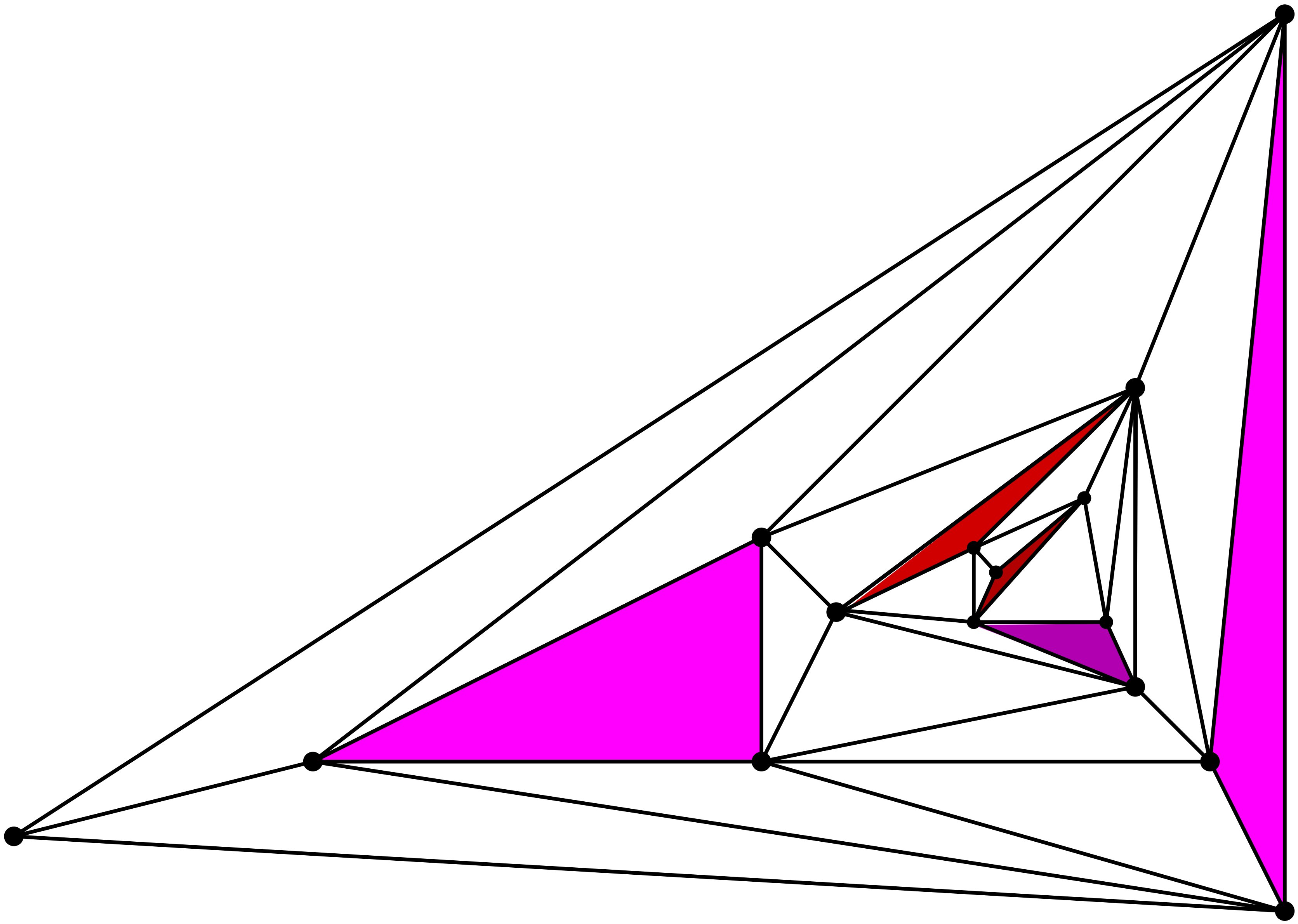

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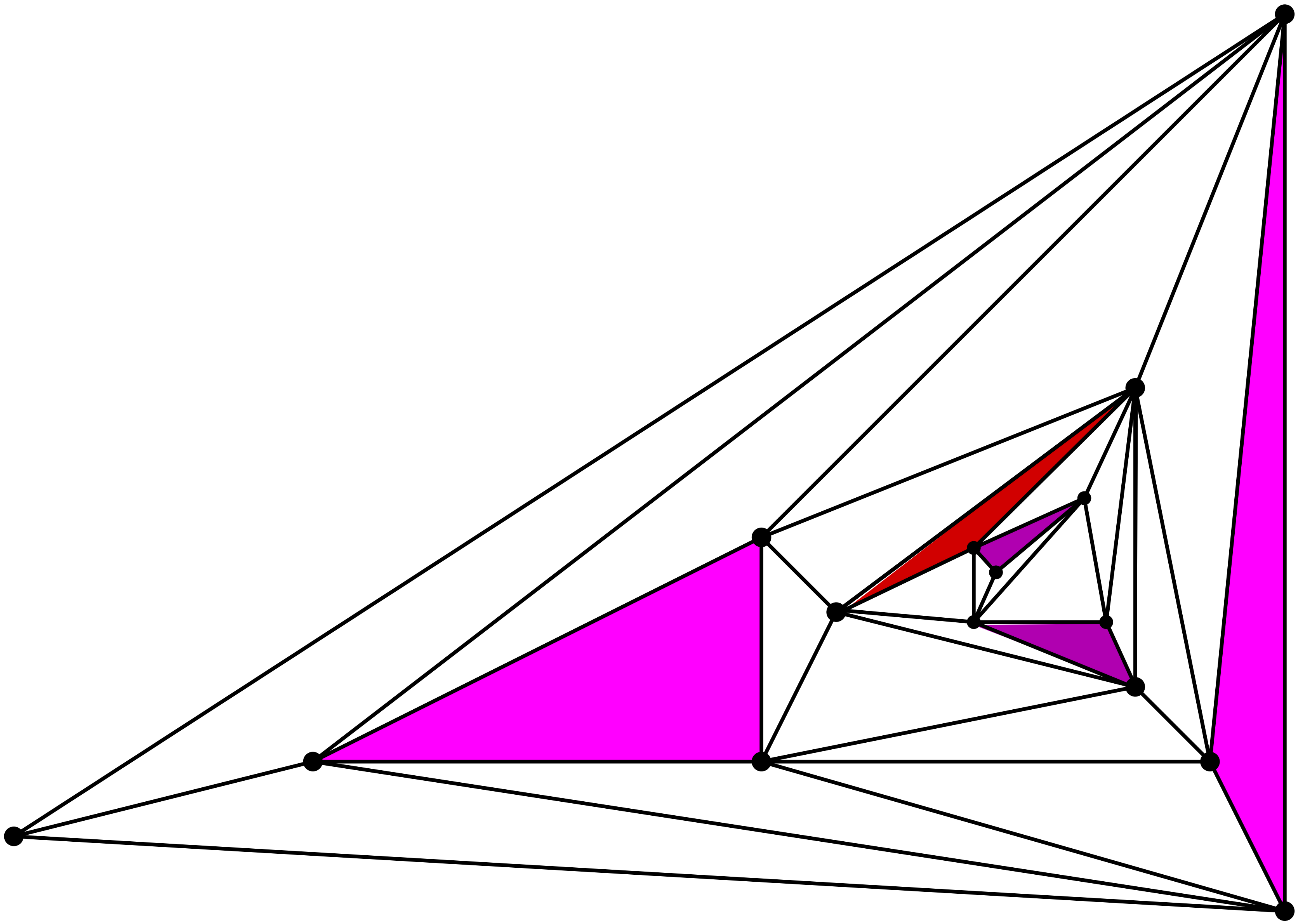

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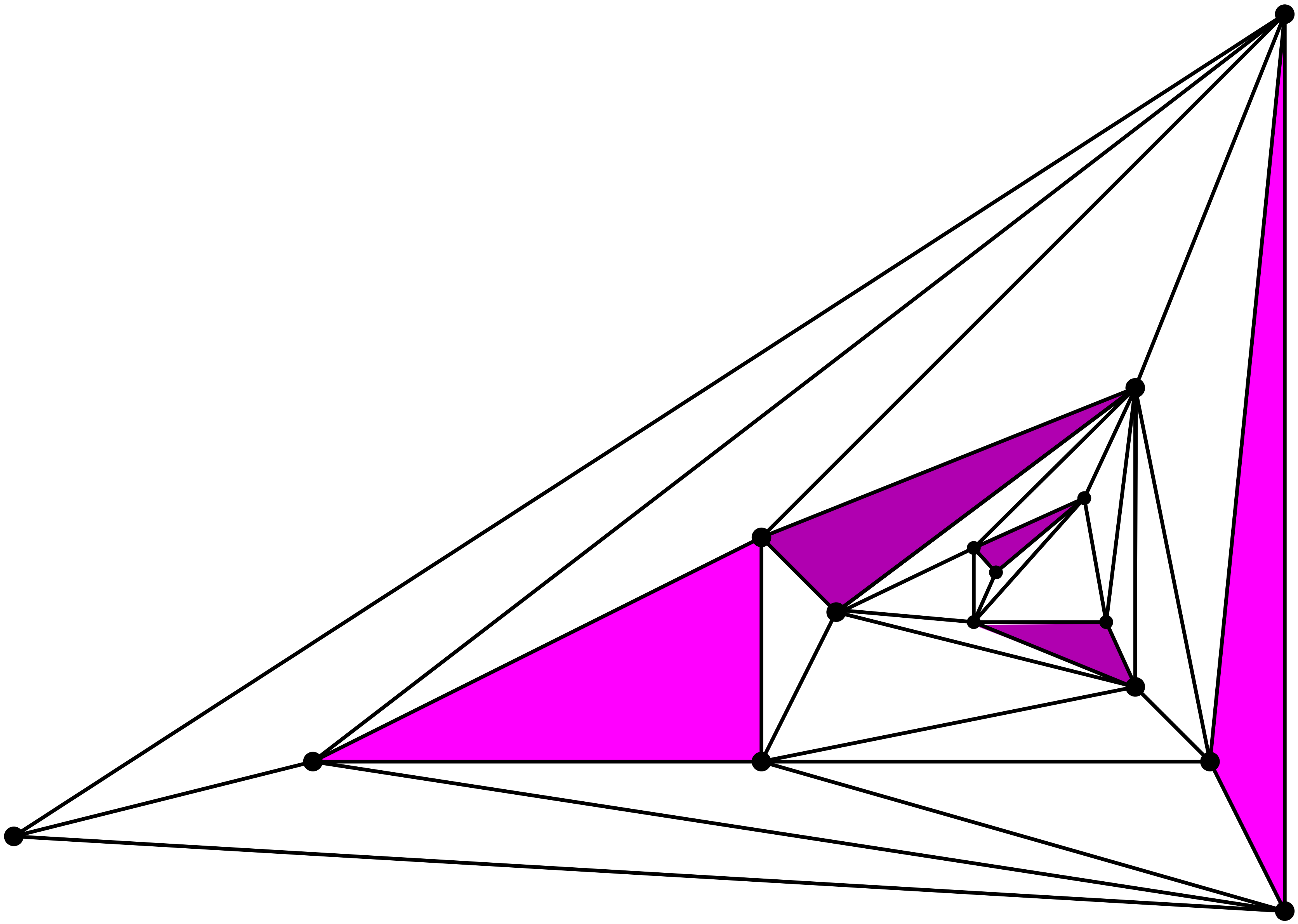

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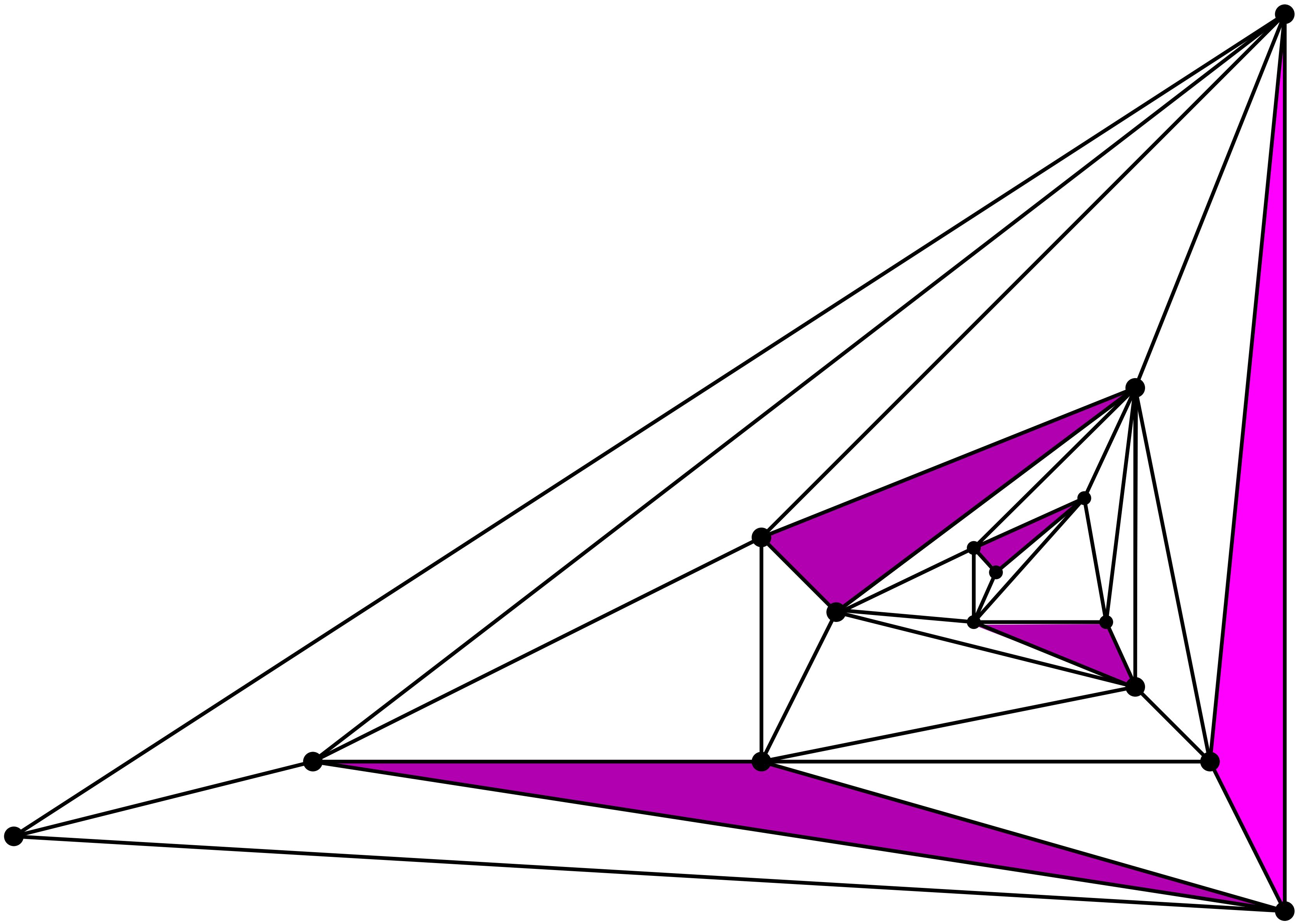

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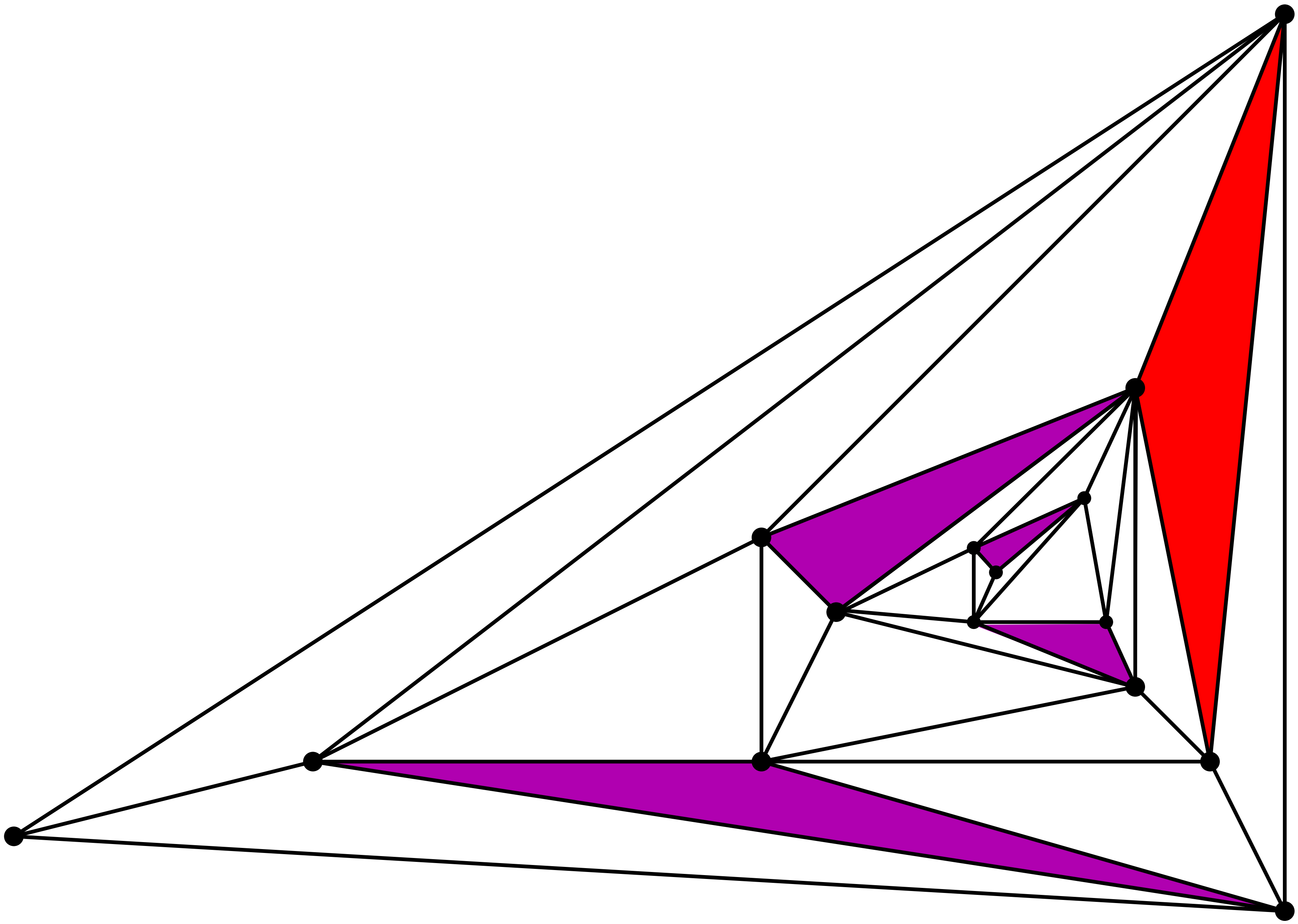

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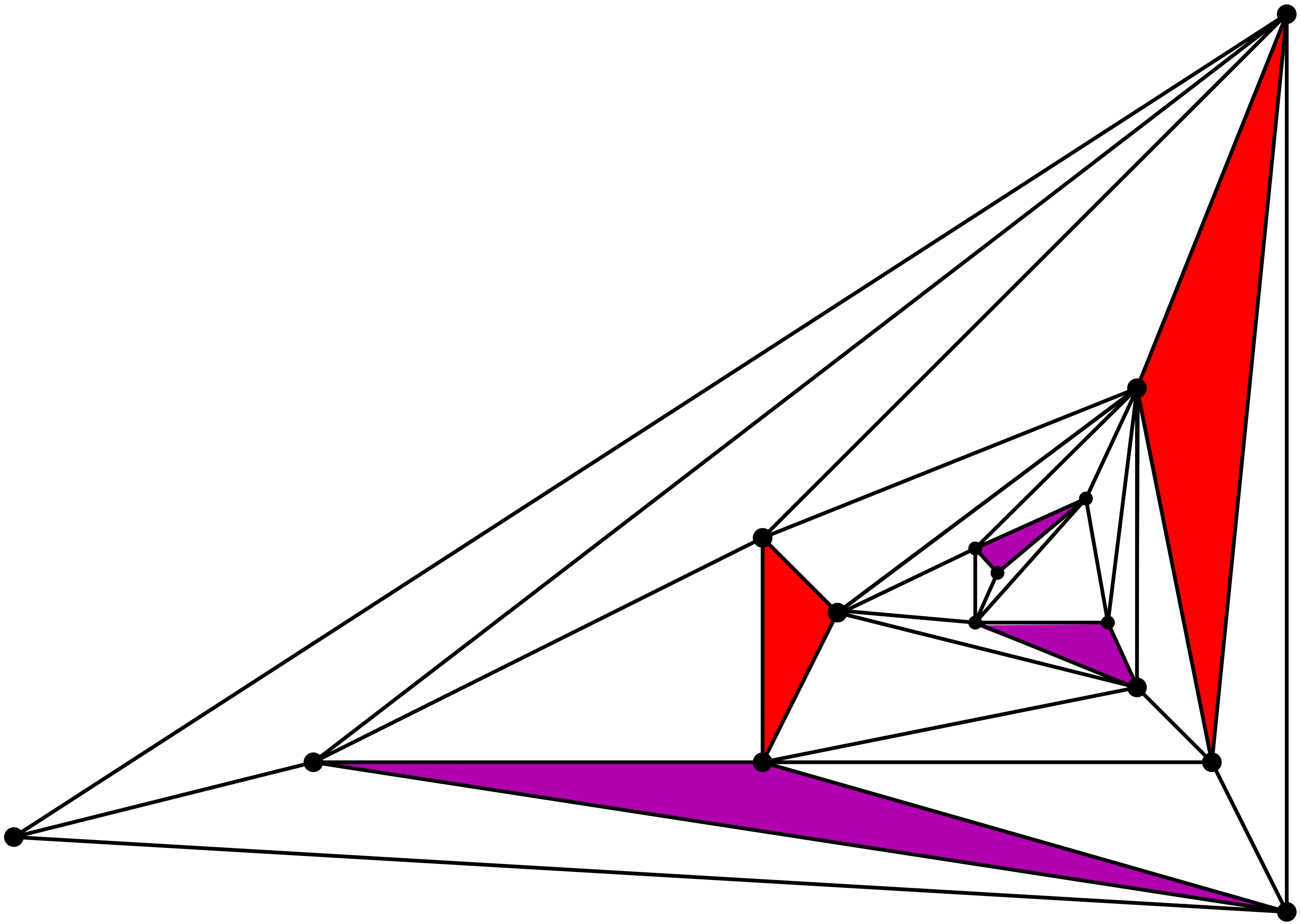

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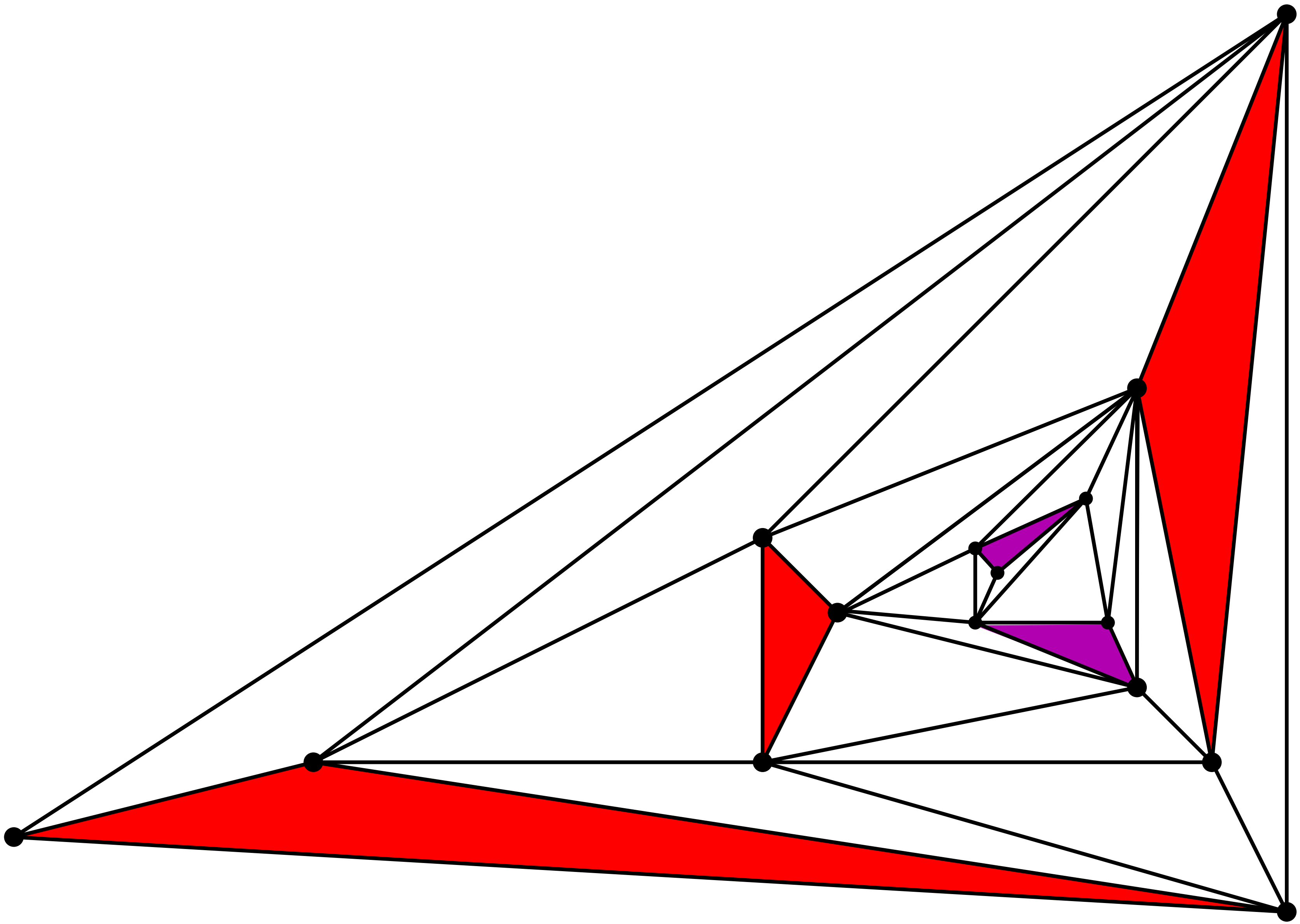


Exponential length of paths

Note:

The path length is exponential in the number of **rooms**!

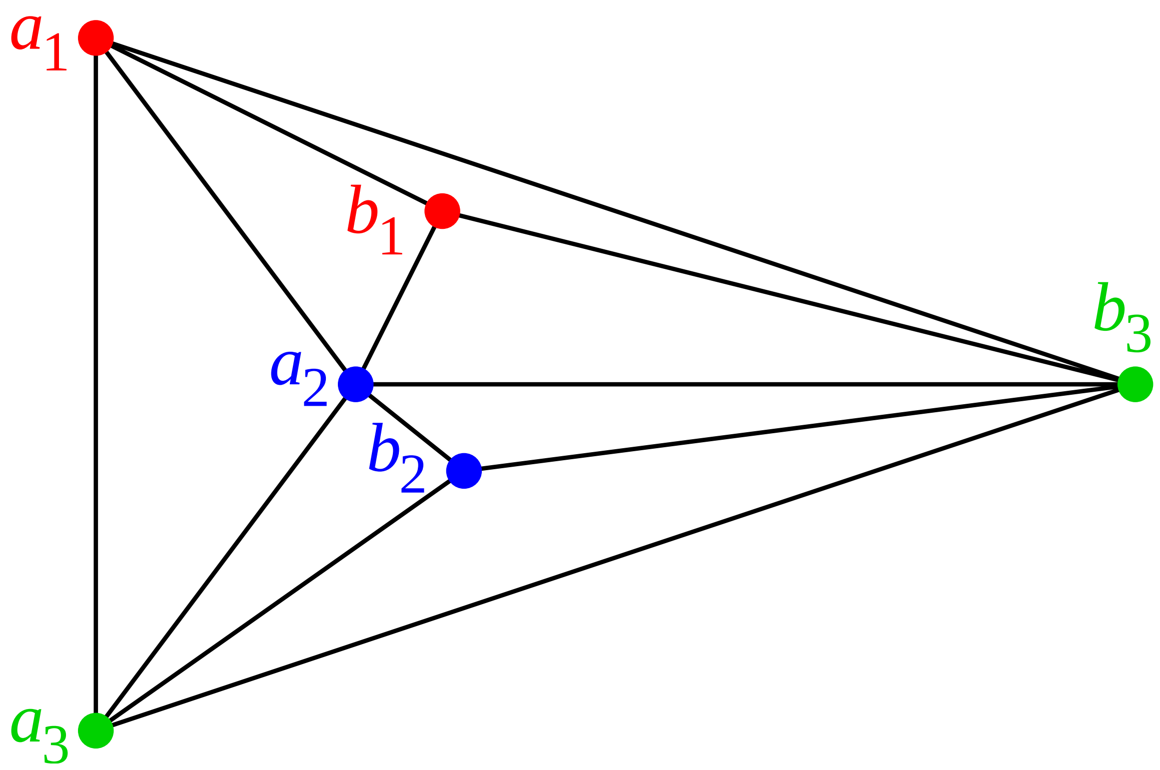
Abstract Sperner

- Manifold M of rank r
- each vertex v has label (color) in {1, . . ., r}
- call a set of vertices **multicolored** if no two of its vertices have the same color.

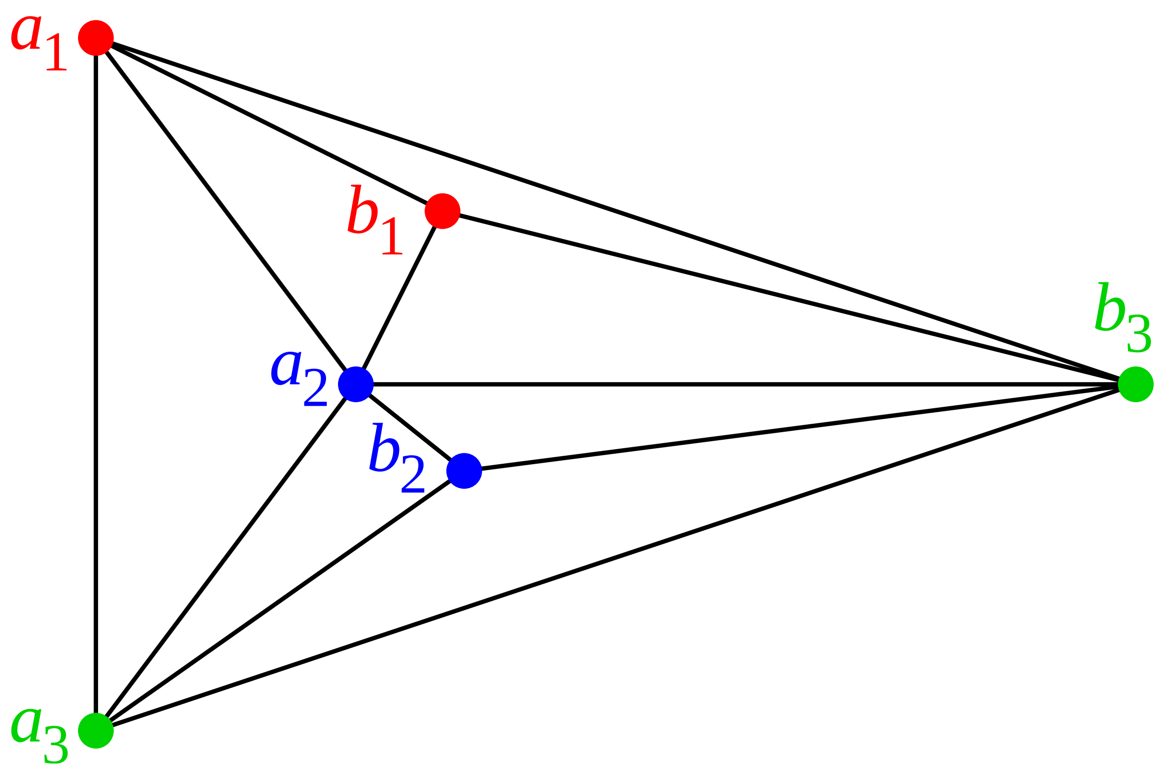
Theorem.

M has an **even** number of multicolored rooms.

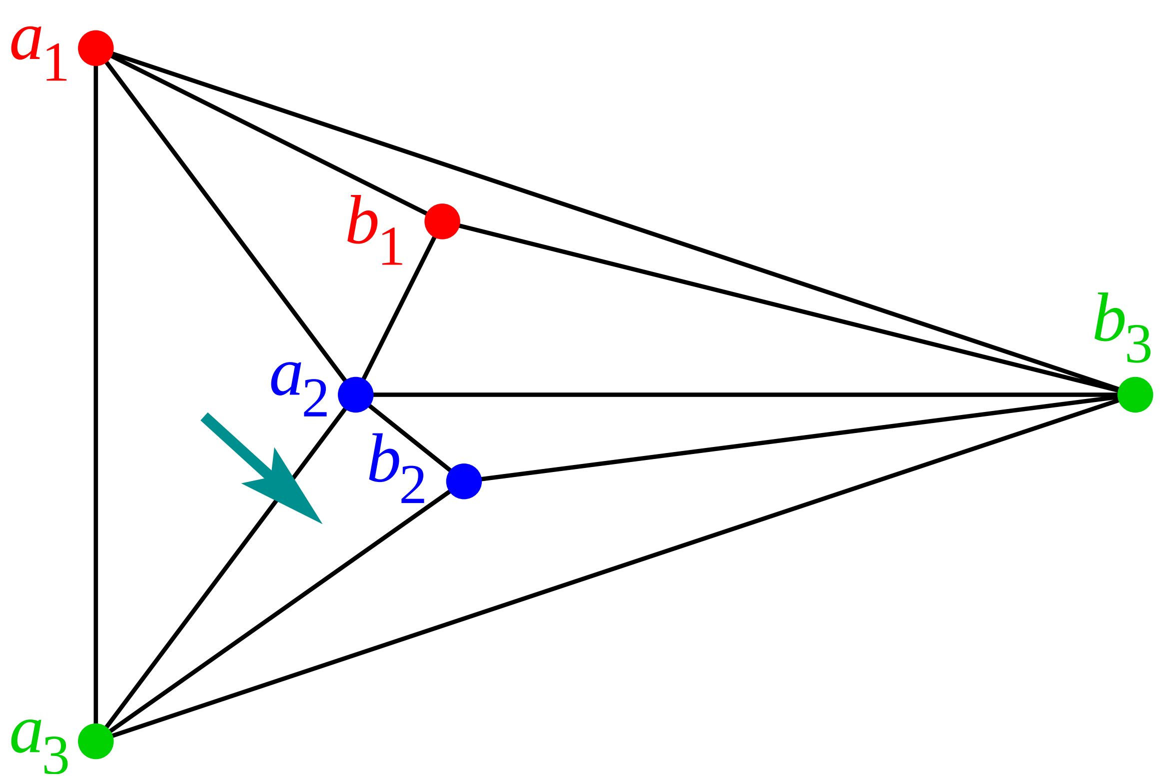
Abstract Sperner



start at multicolored room



allow missing color 1



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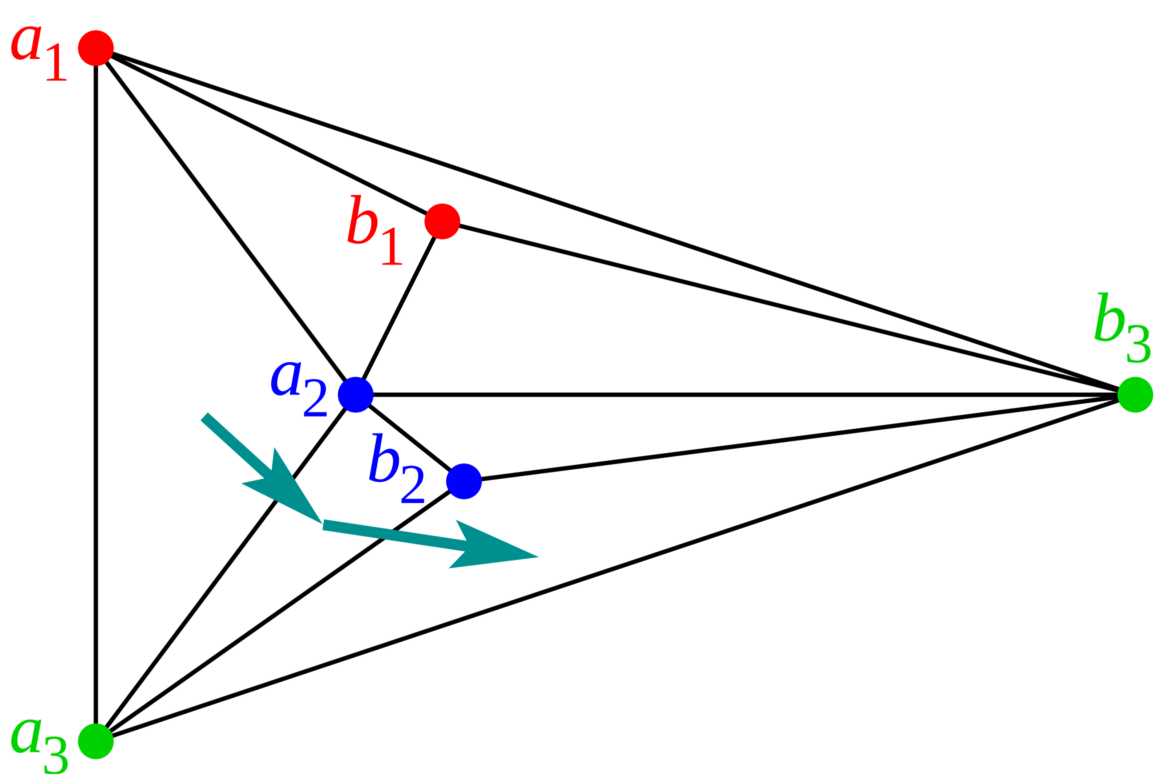

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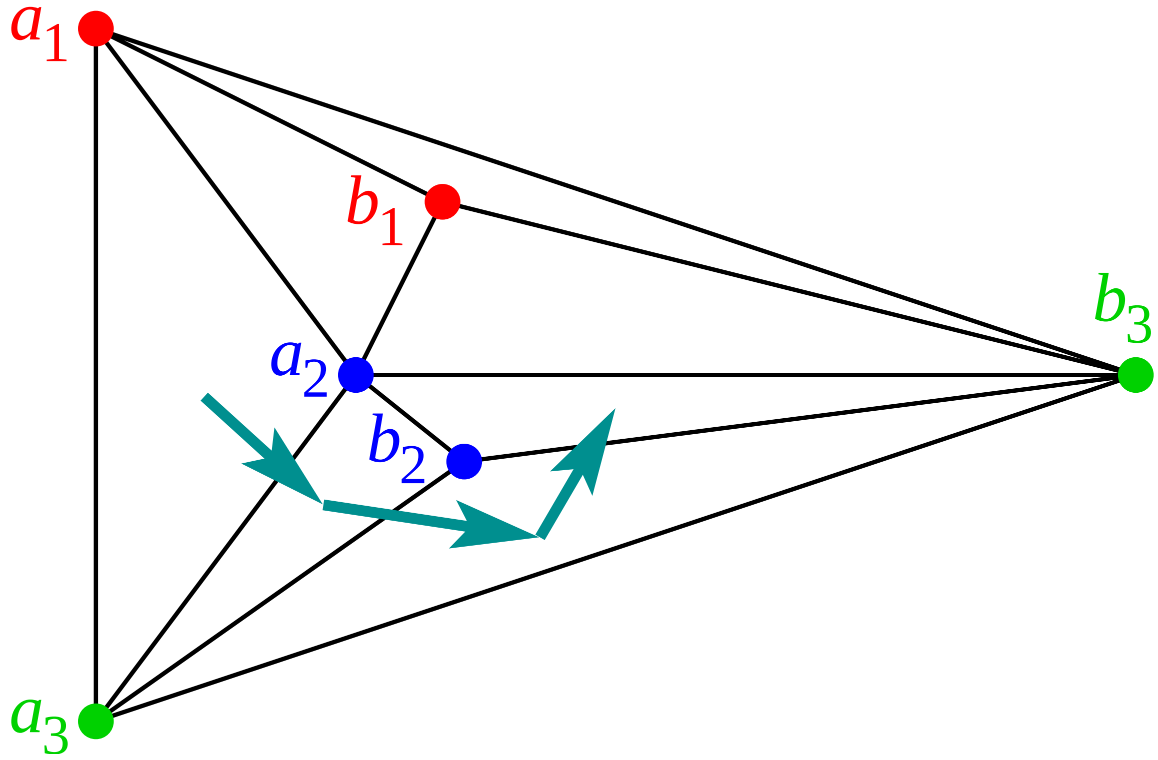
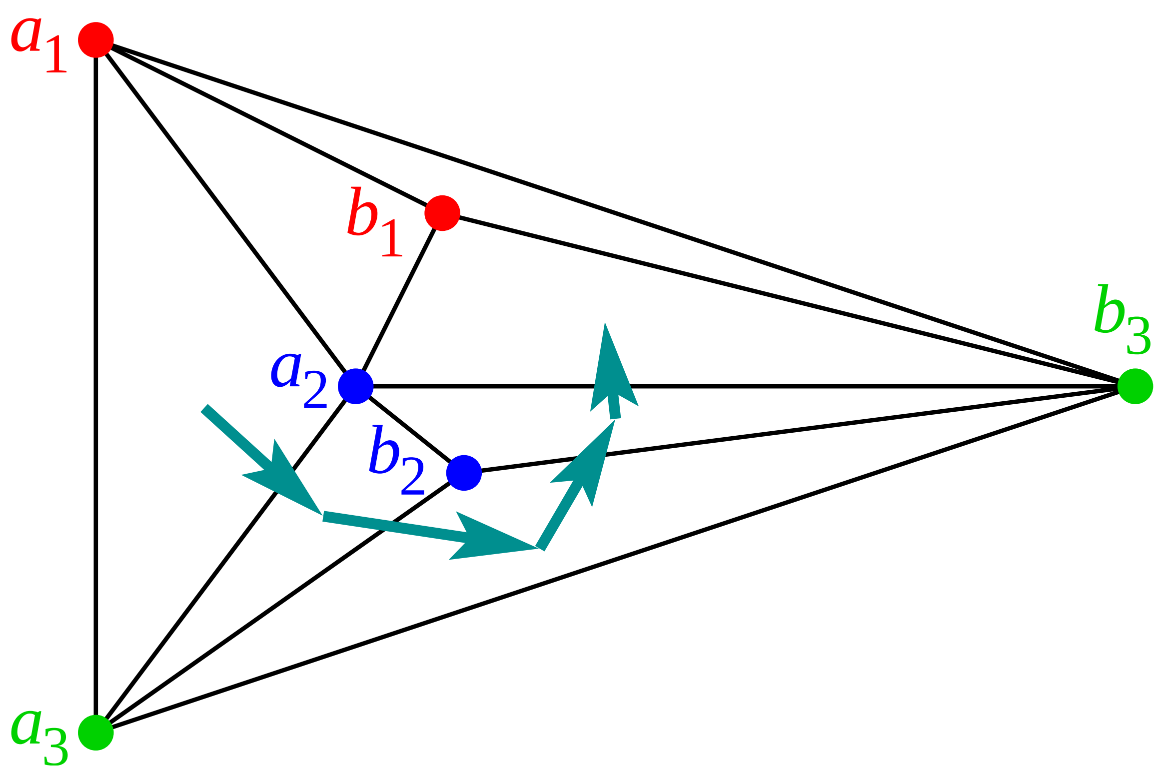


find color 1, done!



Concrete Sperner

- simplex S with r vertices, triangulation T
- each vertex of T has color in {1, ..., r}
- color of a vertex of S **not** found on opposite facet (Sperner condition).

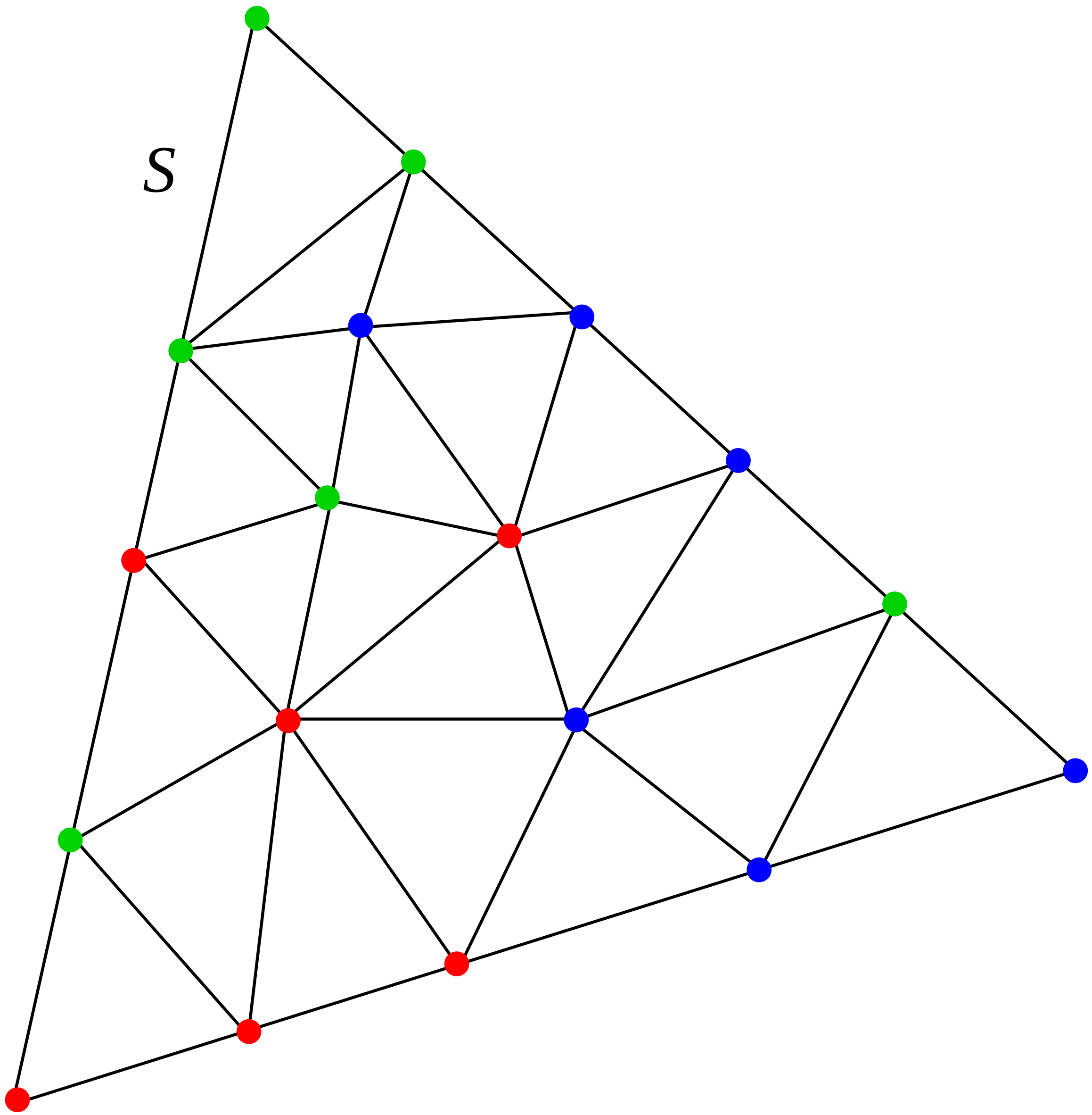
Theorem.

T has an **odd** number of multicolored simplices.

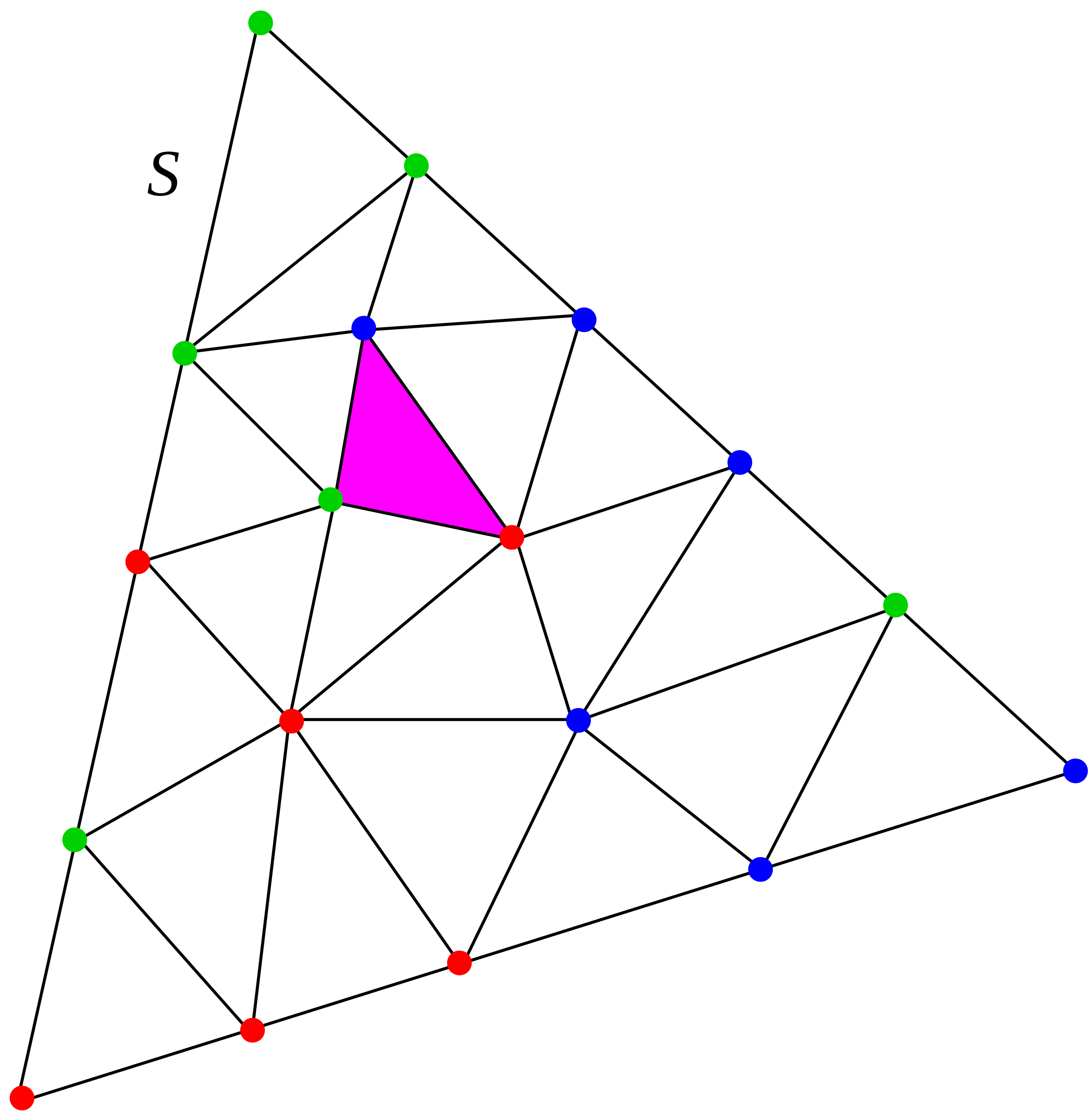
Proof: Corollary to Abstract Sperner

- Induction on rank r: each facet has odd number of multicolored simplices (in one dimension lower)
- Add vertex w of any color and connect w to outside vertices ($\Rightarrow$ get manifold T $\cup$ {w-*rooms*}).
- By induction: T $\cup$ {w-*rooms*} has odd number of multicolored simplices that contain w

$\Rightarrow$ T $\cup$ {w-*rooms*} and hence T has odd number of multicolored simplices that do **not** contain w.



S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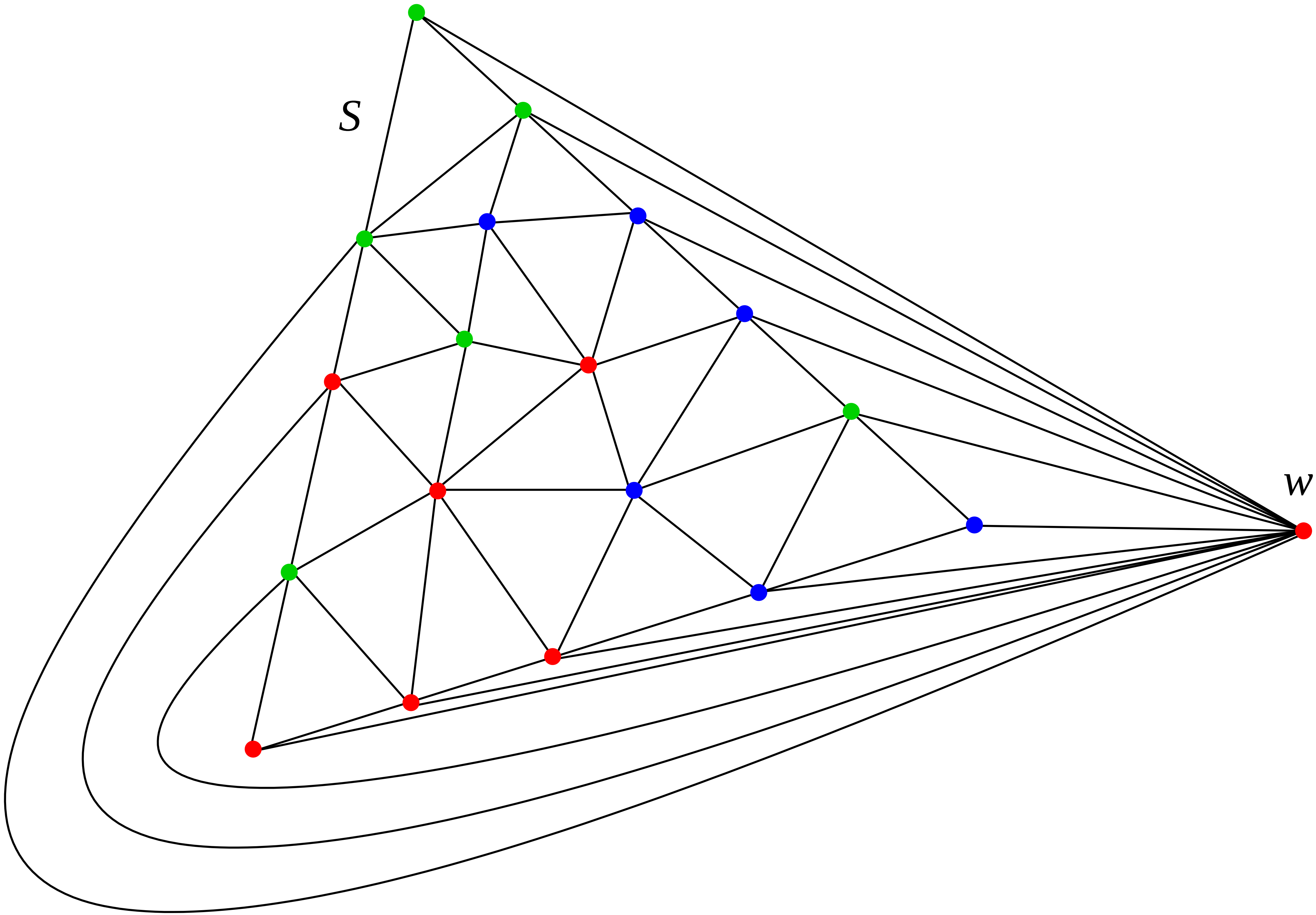
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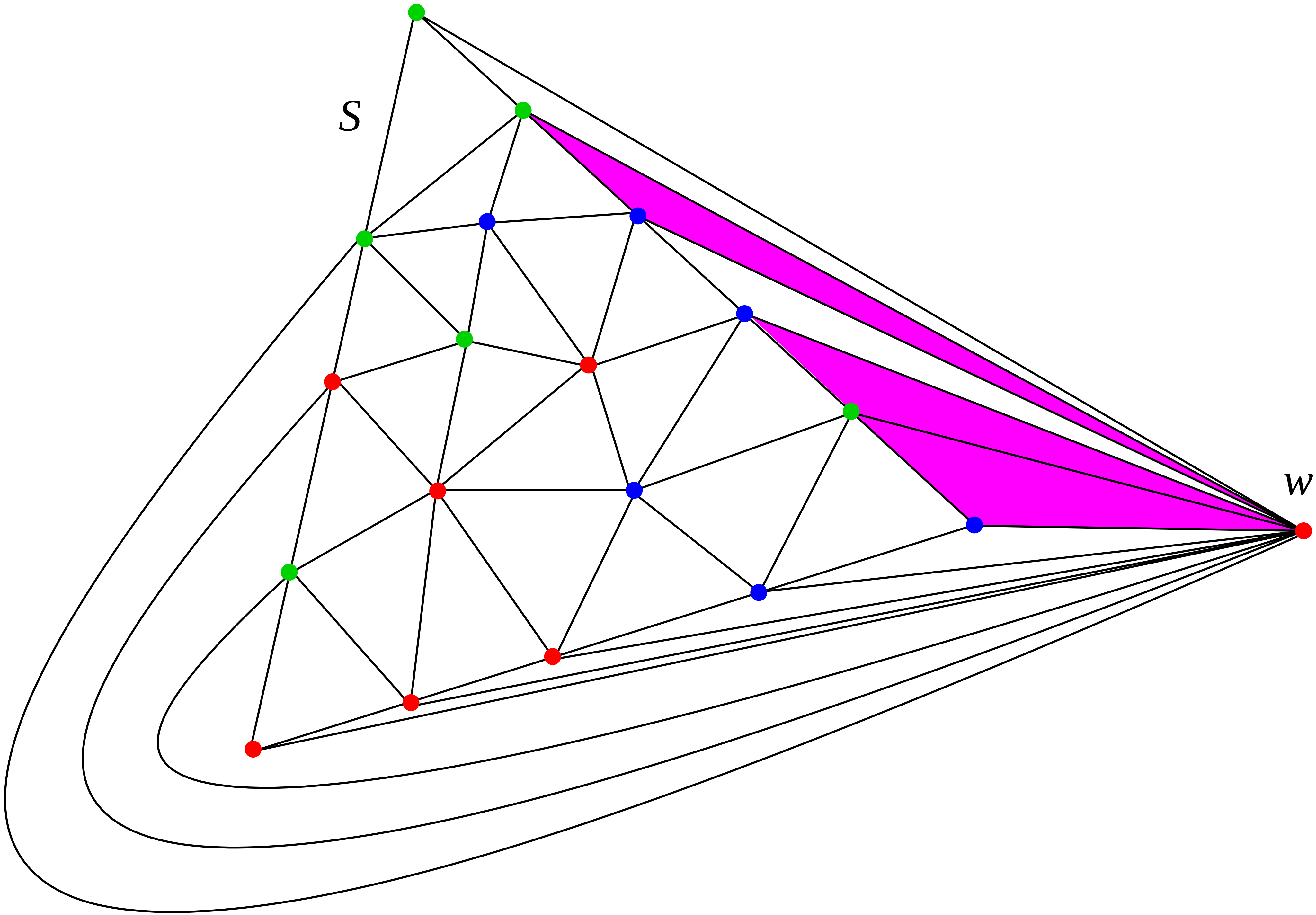
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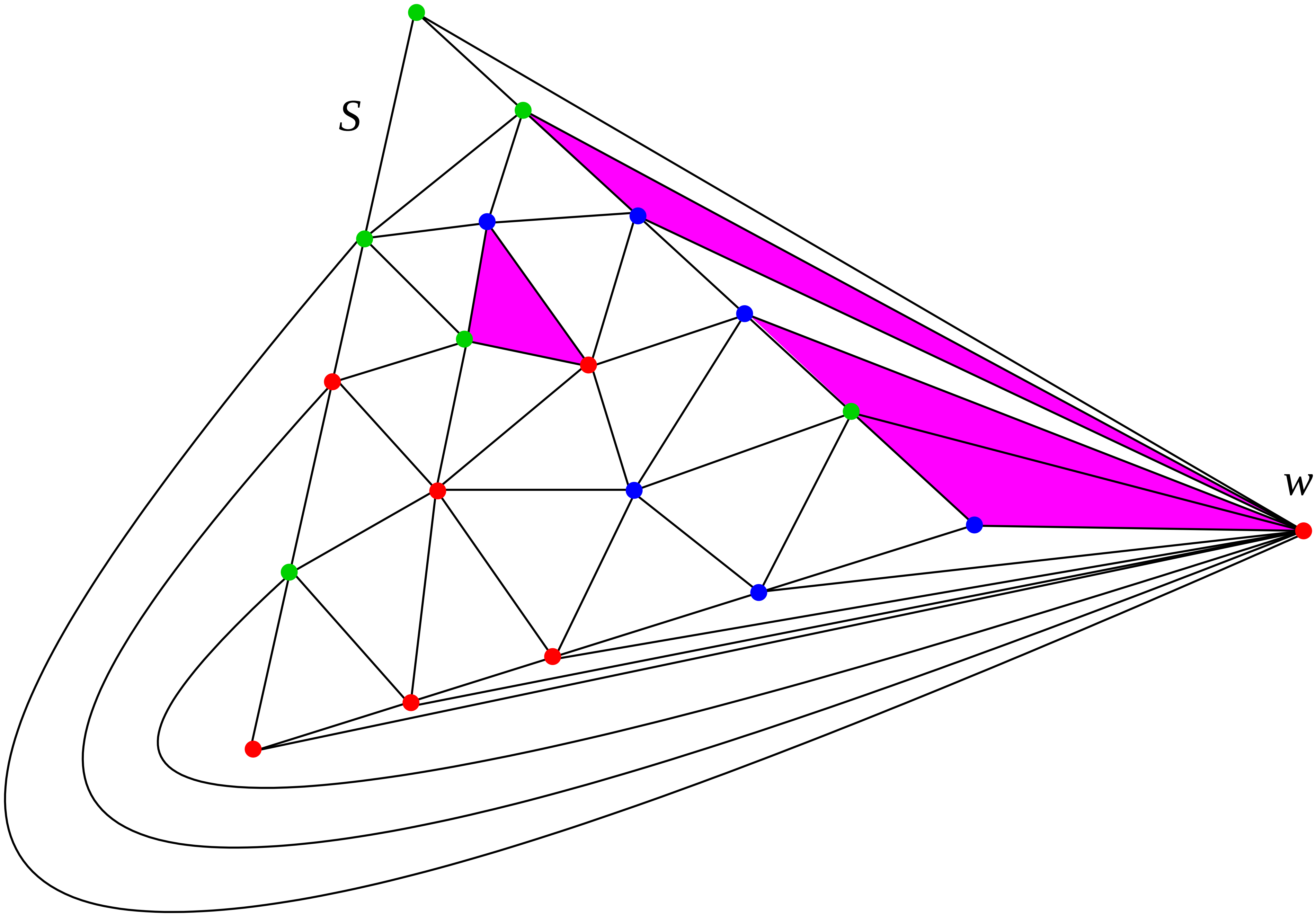
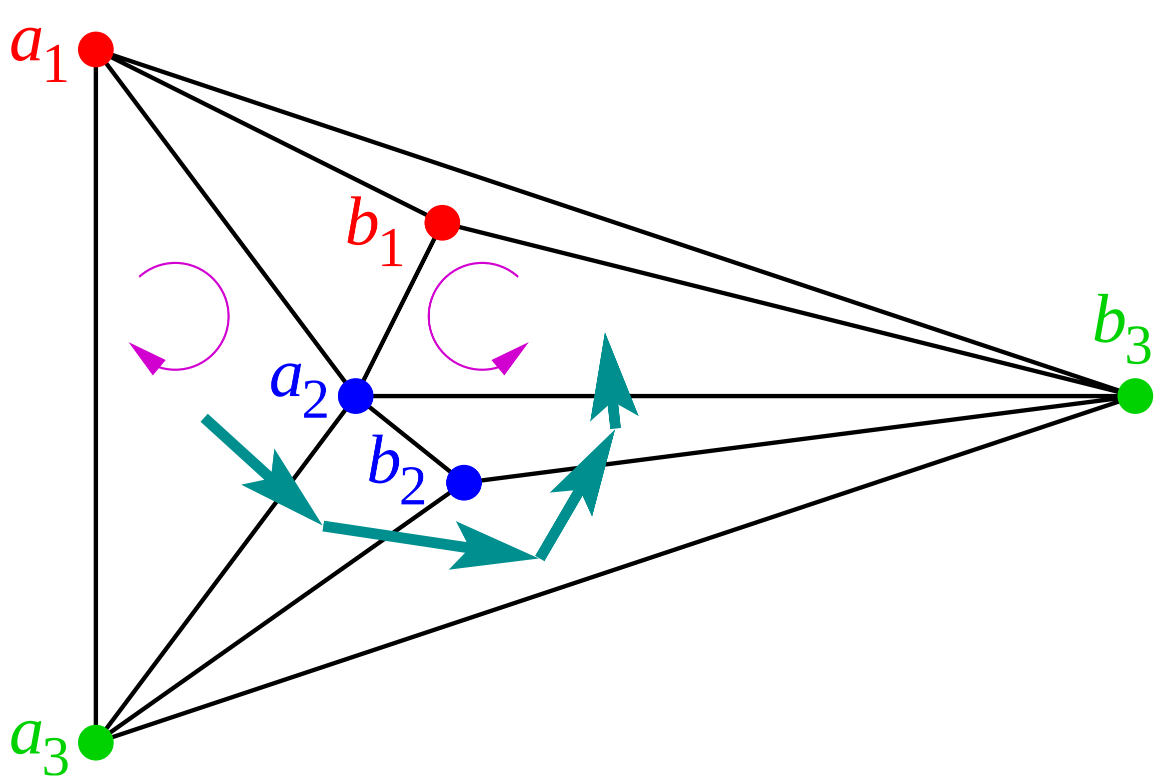

S
w

Sperner and PPAD

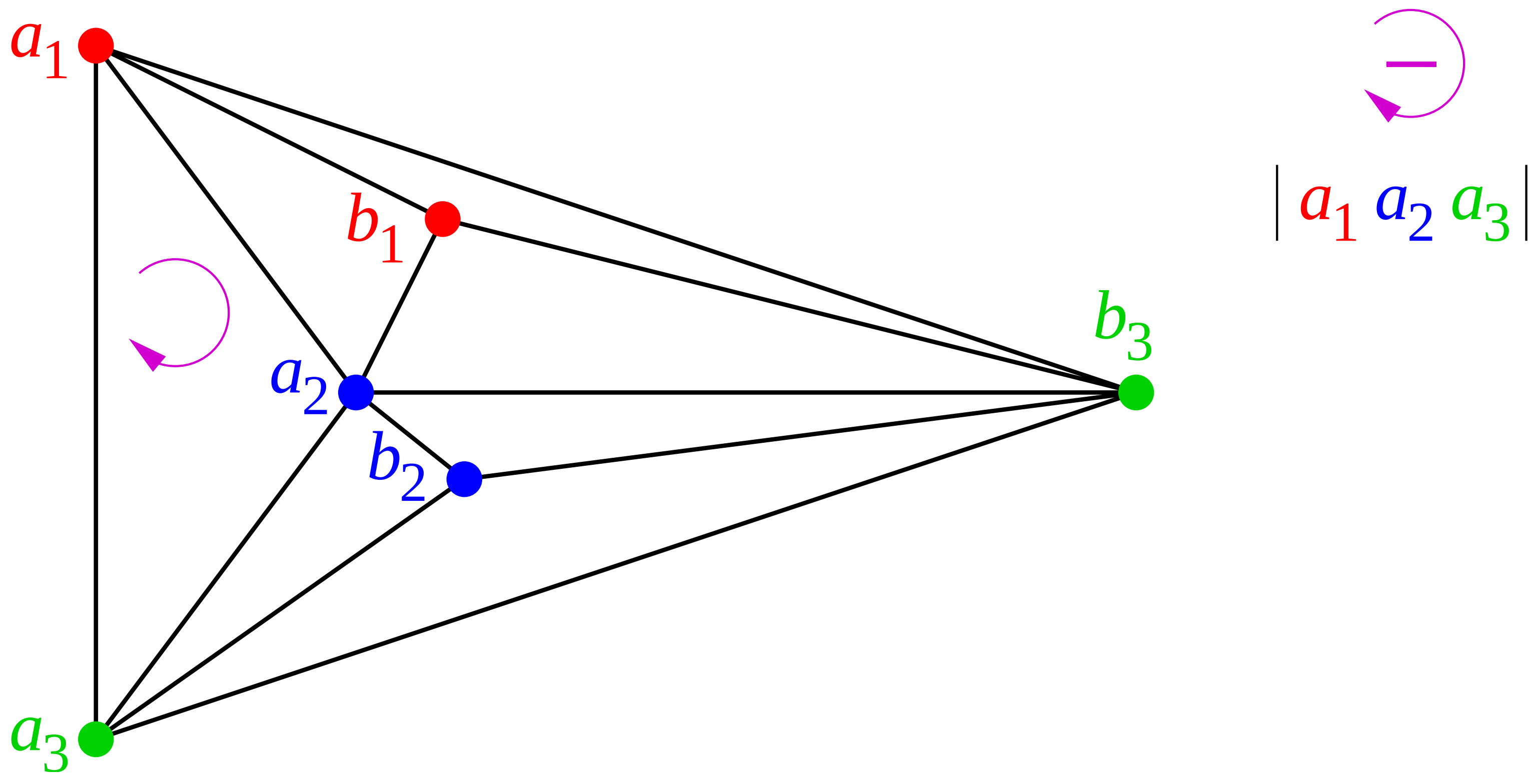
- PPA**D** = proof by parity argument with **direction**,

- direction “local” to the path.

PPAD = opposite orientation of end-rooms



rooms = facets of simplicial polytope

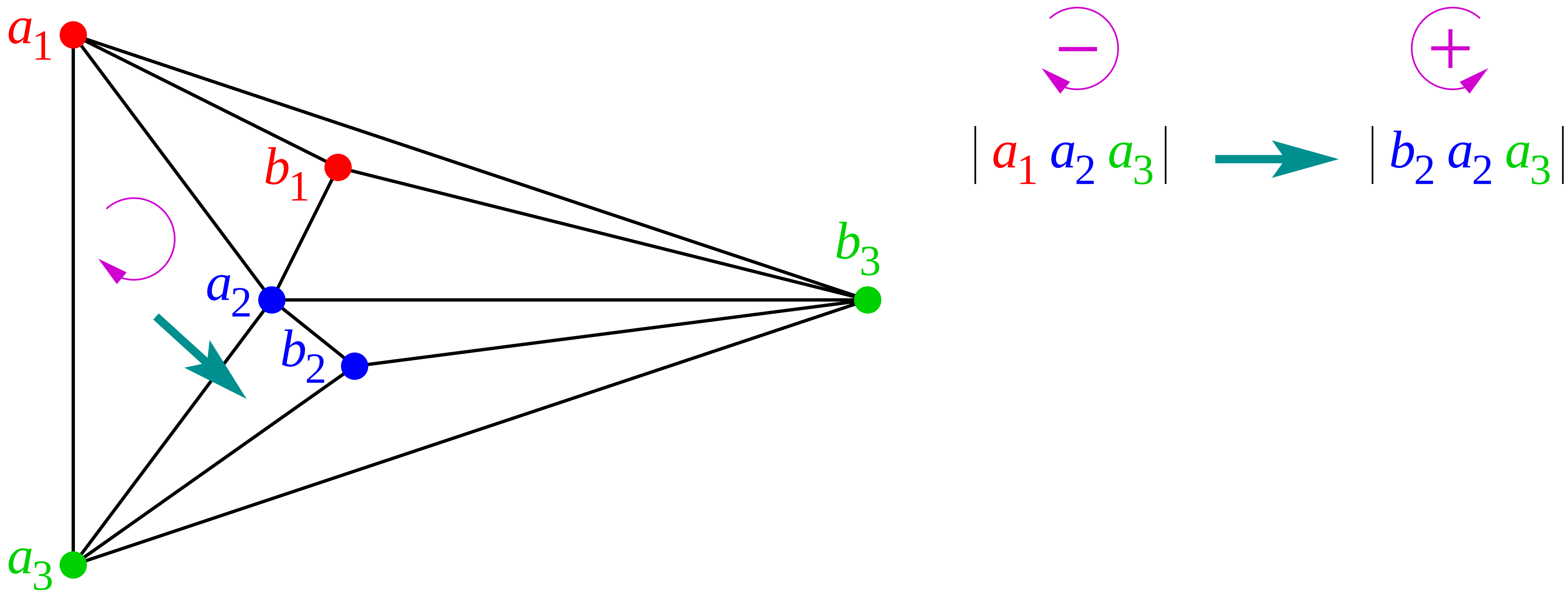


. . . with 0 in interior,

vertices = r-vectors, pivot to next room,

orientation = determinant of vertices in order of labels

rooms = facets of simplicial polytope



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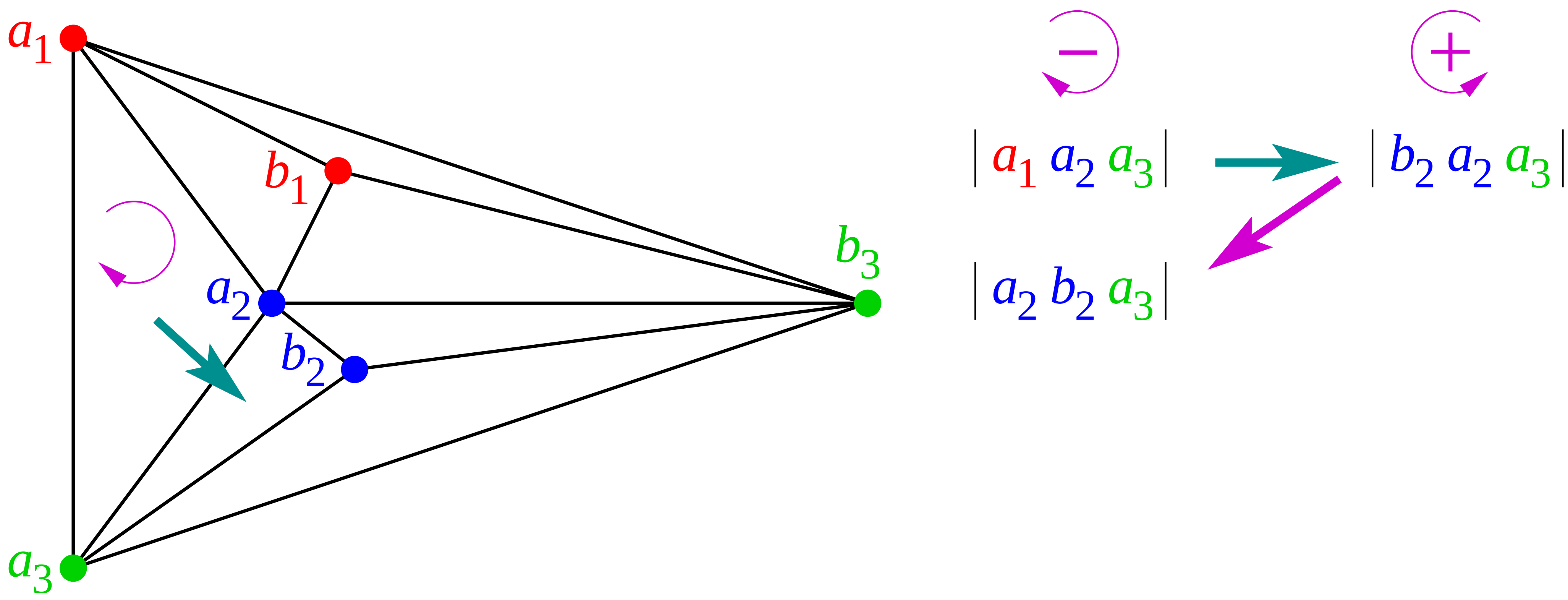

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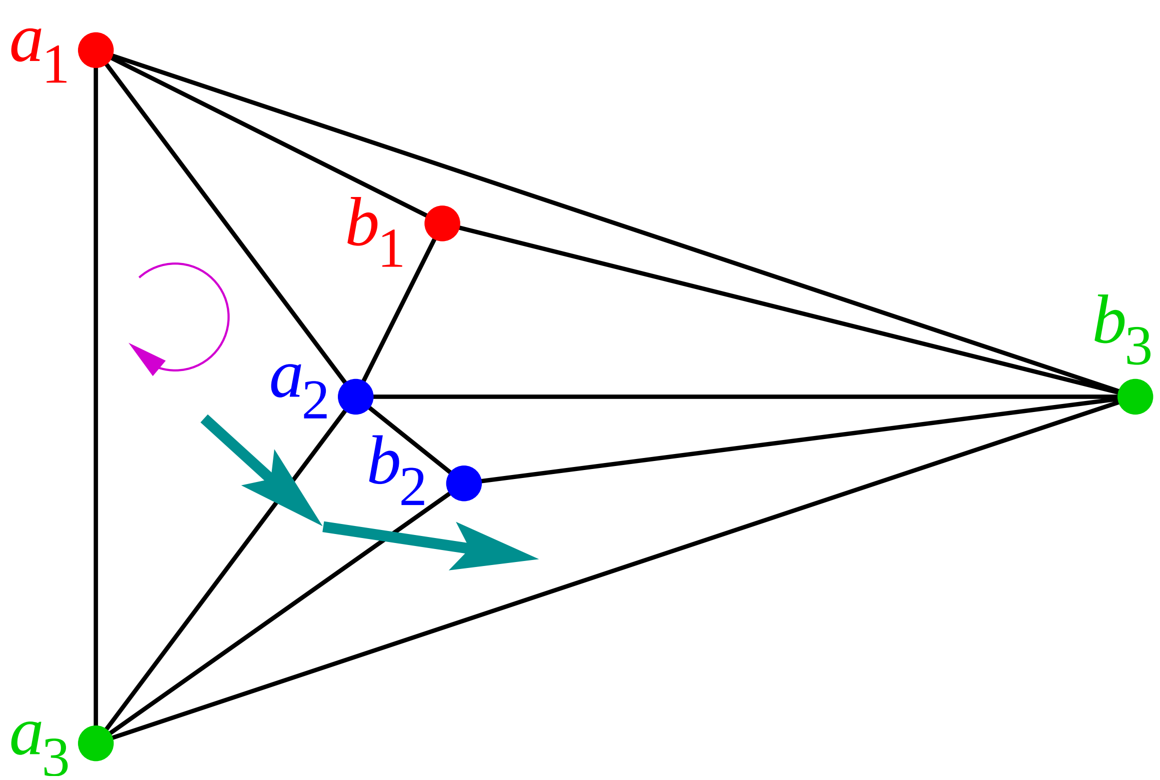

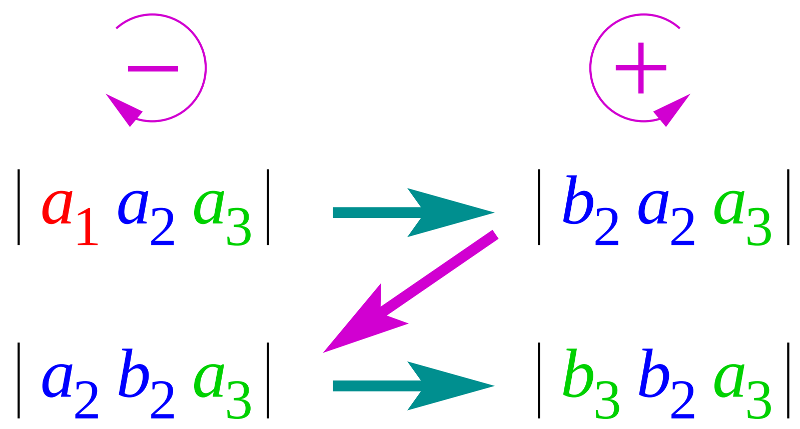

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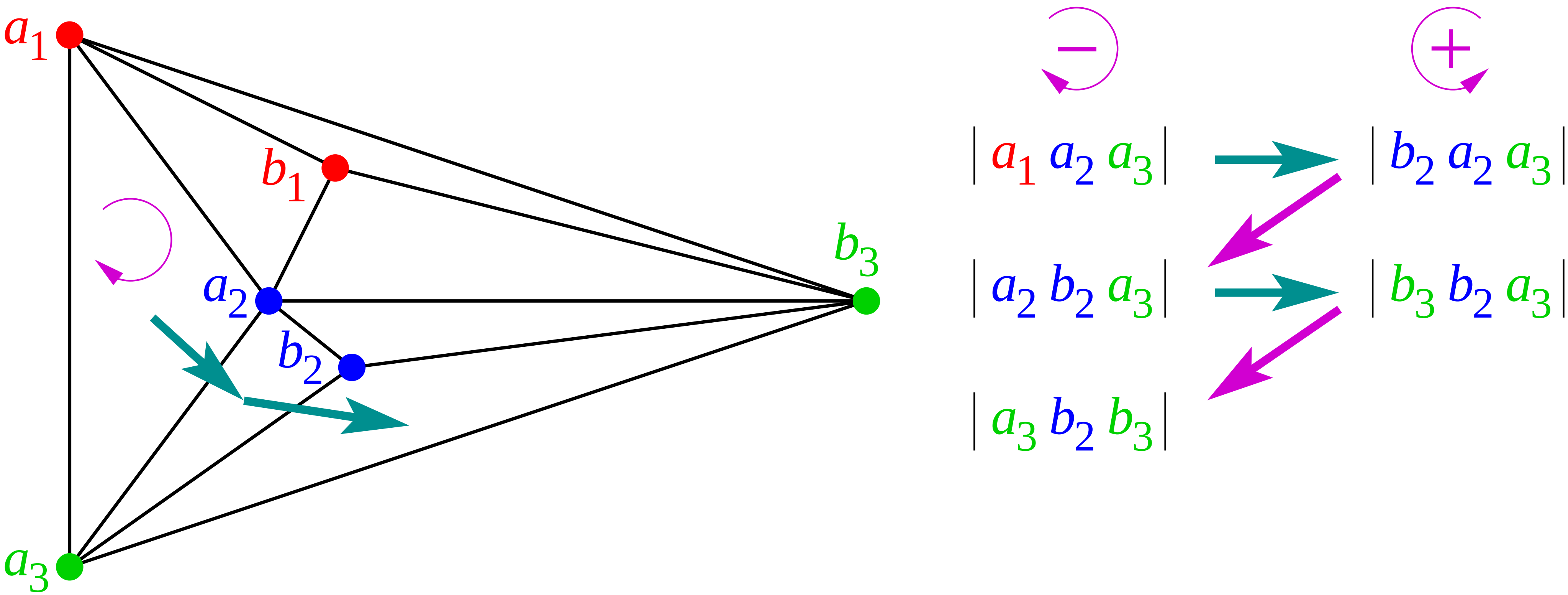

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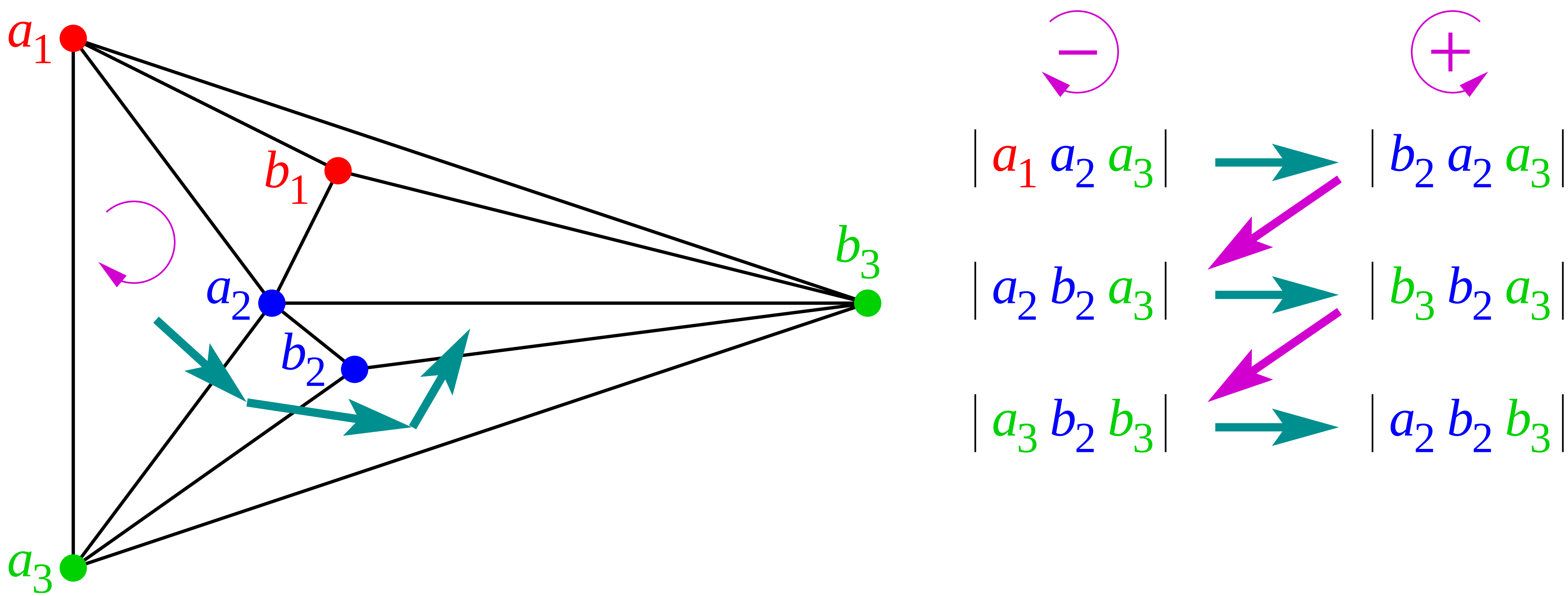

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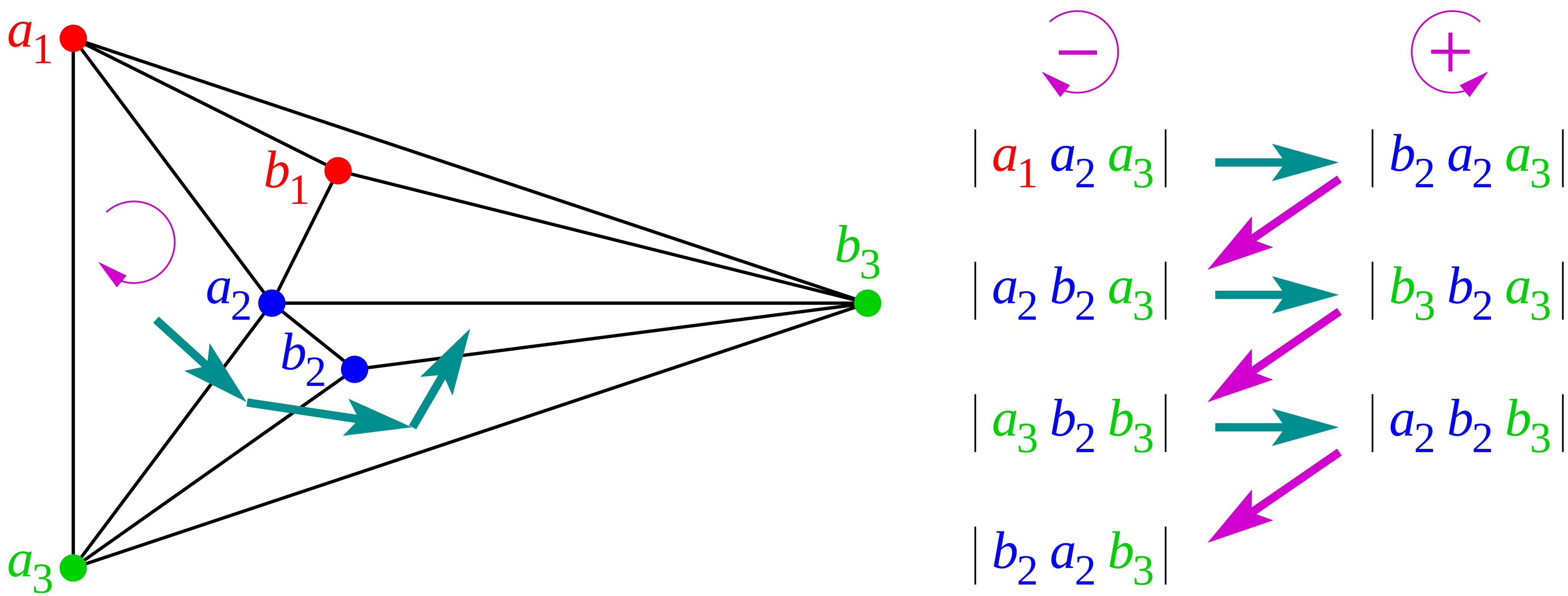

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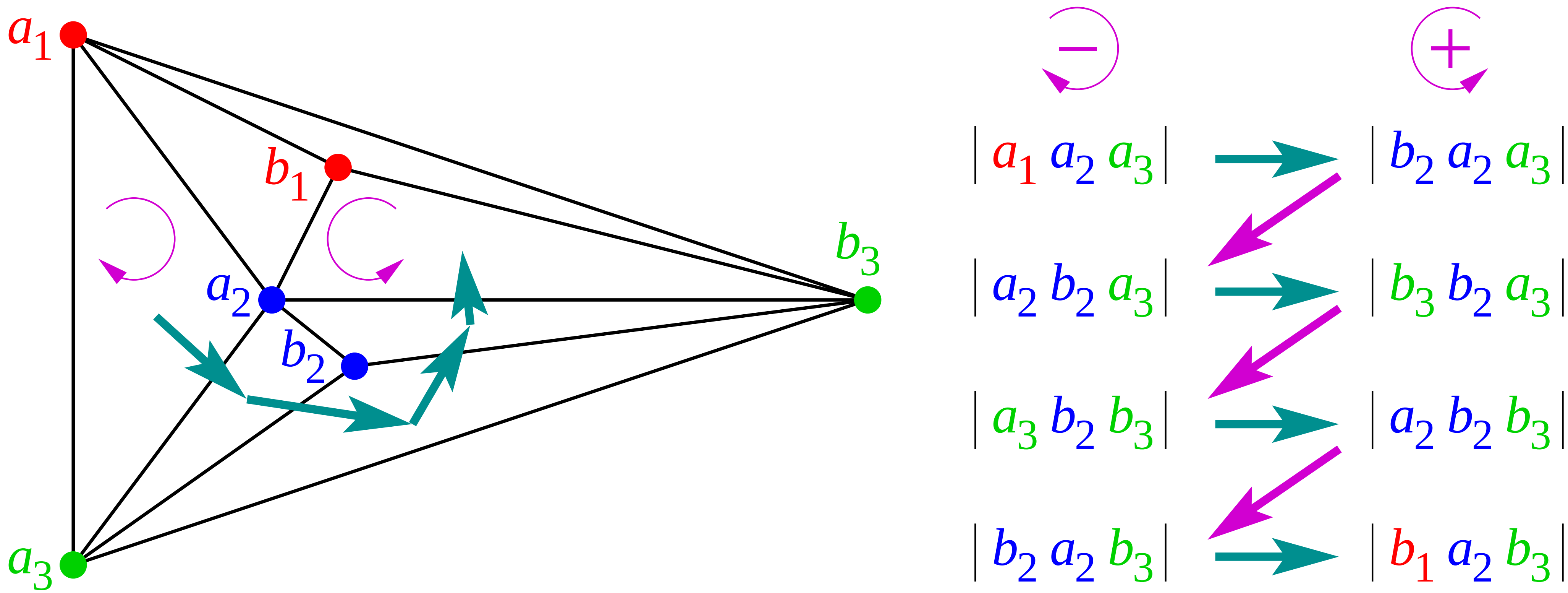

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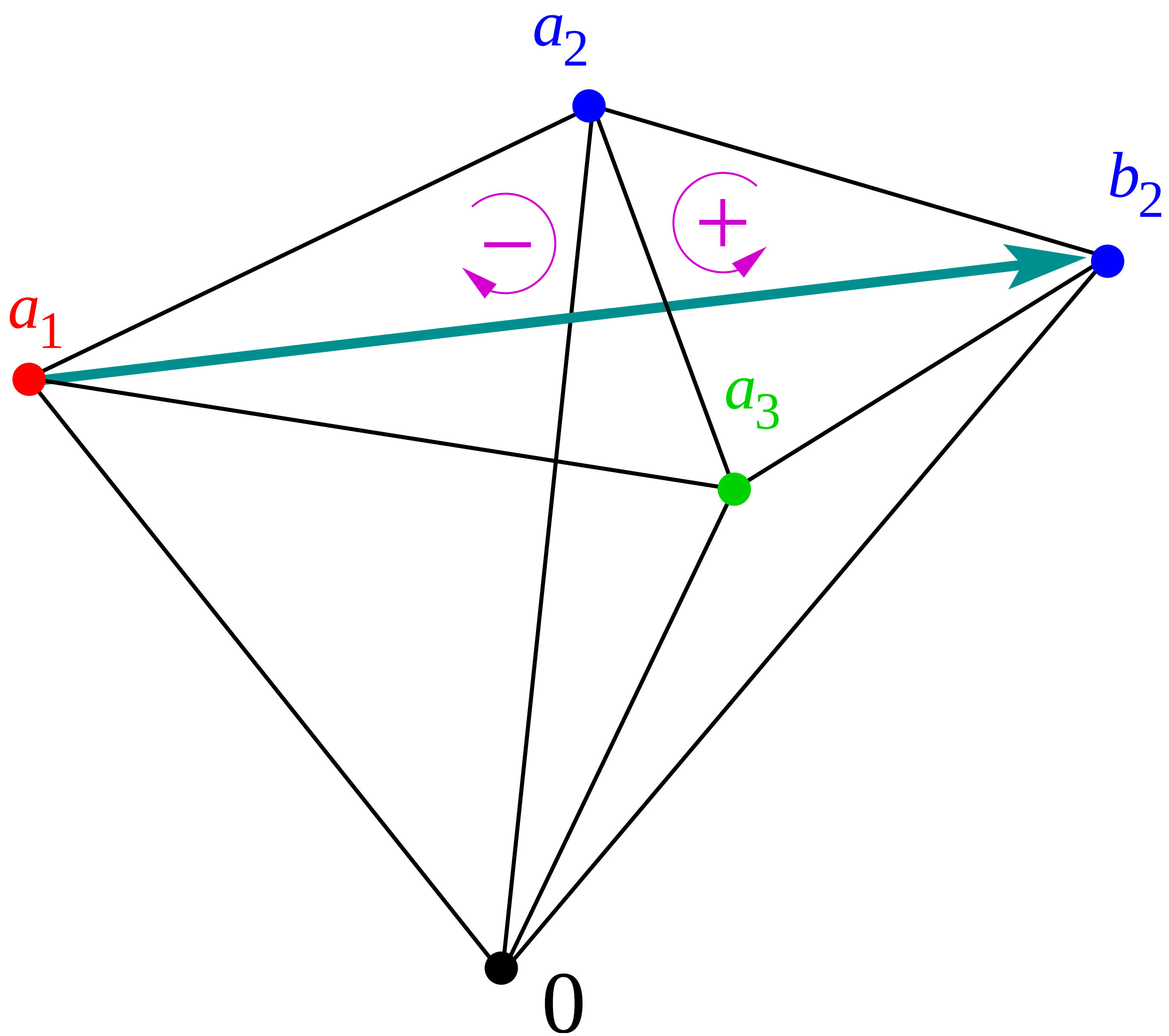
orientation = determinant of vertices in order of labels

Pivoting changes sign of determinant



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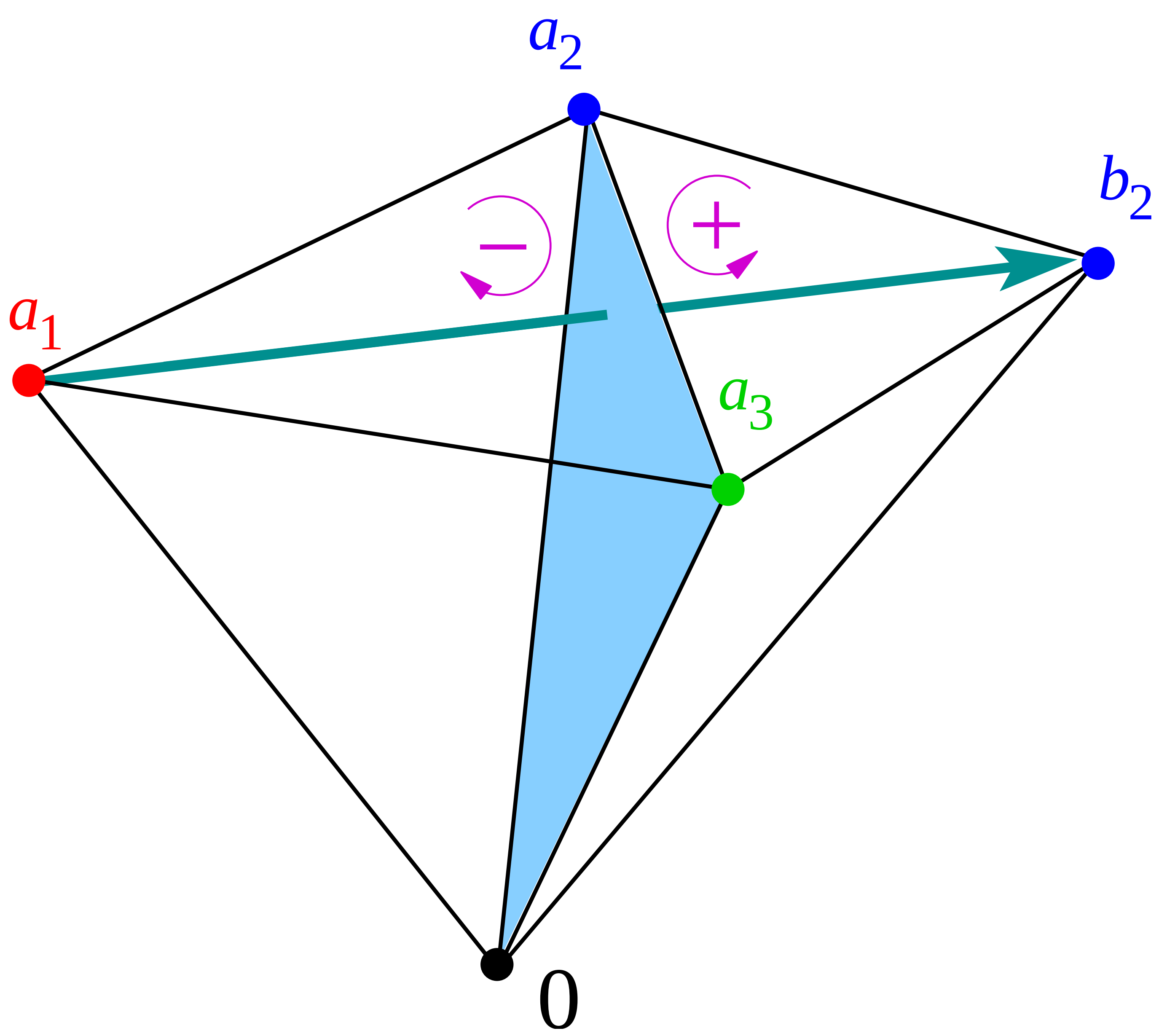

Apply to orientable abstract manifolds

Given a room with vertices $a_1 \, a_2 \, . \, . \, . \, a_r$, call any two **permutations** of the vertices **equivalent** if they differ by an **even** number of transpositions.

$\Rightarrow$ Get two equivalence classes called **orientations**.

Call adjacent rooms $\{a_1 \, a_2 \, . \, . \, . \, a_r\}$ and $\{b_1 \, a_2 \, . \, . \, . \, a_r\}$ **consistently** oriented if $a_1 \, a_2 \, . \, . \, . \, a_r$ and $b_1 \, a_2 \, . \, . \, . \, a_r$ have opposite orientation (doable $\Rightarrow$ oriented manifold).

Simplicial polytopes and games

Given: $M = \text{conv}\{-e_1, \ldots, -e_r, b_1, \ldots, b_n\}$ with labels

- i for negative unit vector $-e_i$, $i = 1,\ldots,r$
- $c(j) \in \{1, \ldots, r\}$ for r-vector $b_j > 0$, $j = 1,\ldots,n$.

Then: completely labeled facets of M

$\Leftrightarrow$ Nash equilibria of the $r \times n$ bimatrix game

$([\, e_{c(1)} \ldots e_{c(n)} \,], [\, B_1 \ldots B_n \,])$

where $B_j = b_j / (1 + \| b_j \|_1)$, $j = 1,\ldots,n$.

Path lengths for Abstract Sperner

- Linear in number of rooms.
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  - $*$ if rooms = facets of simplicial polytope,

    path-following via pivoting.

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  rooms = Gale evenness bit-strings (1 = vertex)

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  (0's and even runs of 1's).

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Example [Morris 1994]

c = 123456645231

111111000000 ... 000110110110
011111100000 000011110110
011110110000 000011011110
011011110000 000001111110
011011011000 000000111111
011001111000
011000111100
001100111100 In general: N = 2r,
001101101100
001111001100 path exponentially
001111000110
000111100110 long in r .

Open problems

Is “**Find a second room partitioning**” in PPAD?

[Direction at most possible when choosing missing vertex *w*, otherwise don't get opposite orientations of partitions at end of paths even for polytopes, e.g. octahedron.]

... **PPA-complete**? [No such problem known.]