Tight Formulations for Some Simple Mixed Integer Sets

Laurence A. Wolsey

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Two Variable MIP Set

$$X = \{(z, s) \in Z^1 \times R^1_+ : s + z \geq b\}$$

where $f = b - \lfloor b \rfloor > 0$.

The mixed integer rounding (MIR) inequality

$$s \geq f(\lfloor b \rfloor + 1 - z)$$

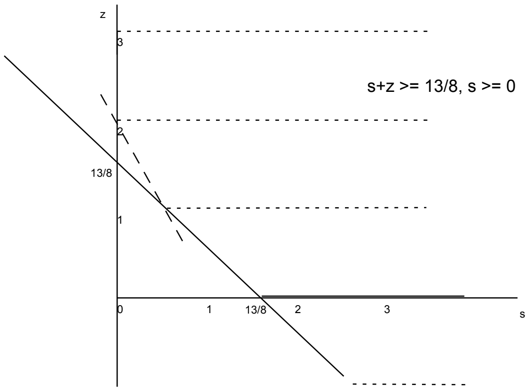
is valid for X.

$X = \{(z, s) \in Z \times R^1_+ : s + z \geq \frac{13}{8}\}$
MIR Inequality: $s \geq \frac{1}{3}(z - 3)$

MIR Example

$X = \{(z, s) \in Z \times R^1_+ : s + z \geq \frac{13}{8}\}$

MIR Inequality: $s \geq \frac{5}{8}(2 - z)$



Questions to be asked about a “simple" set X

- Describe a family $\mathcal{F}$ of valid inequalities for X or $\mathrm{conv}(X)$
- Describe a separation algorithm for $\mathrm{conv}(X)$, or for the polyhedron described by the family $\mathcal{F}$
- Describe an extended formulation for X, if possible providing a tight formulation of $\mathrm{conv}(X)$
- (Additional Constraints). Given X, suppose that $P = \mathrm{conv}(X)$.
  Now consider $X \cap Q$ where Q is a polyhedron.
  For which polyhedra Q is it true that

$$\mathrm{conv}(X \cap Q) = \mathrm{conv}(X) \cap Q?$$

MIR gives Convex Hull: Proof

$$\begin{aligned} s + z &\geq b \\ s + fz &\geq f\lceil b \rceil \\ s &\geq 0 \end{aligned}$$

In a facet, $s = \max(b - z,\ f(\lceil b \rceil - z),\ 0)$.

- If $s = b - z$, then facet is $z \leq \lfloor b \rfloor$.
- If $s = f(\lceil b \rceil - z)$, then facet is $\lfloor b \rfloor \leq z \leq \lceil b \rceil$
- If $s = 0$, then facet is $z \geq \lceil b \rceil$.

Generalization

$$X^* = \{(s,z) \in \mathbb{R}^n_+ \times \mathbb{Z}^n : s^i + z^i \geq b_i, i = 1, \ldots, n\}.$$

Convex hull is obtained by adding MIR inequalities.

When is convex hull of $X^* \cap Q$ given by MIR inequalities?

When $Q = \{z \in \mathbb{R}^n : Dx \leq d\}$ with D totally unimodular and d integer.

Multi-Item Discrete Lot-sizing Problem

$$\min \sum_{i=1}^{NI} \sum_{t=1}^{NT} (h_t^i s_t^i + b_t^i r_t^i + q_t^i y_t^i)$$
$$s_{t-1}^i - r_{t-1}^i + C^i y_t^i = d_t^i + s_t^i - r_t^i \ \forall i, t$$
$$\sum_{i=1}^{NI} y_t^i \leq 1 \ \forall t$$
$$s_t^i, r_t^i \geq 0, y_t^i \in \{0, 1\} \ \forall i, t$$

Equivalent formulation of flow constraints
Eliminate the variables r_t giving

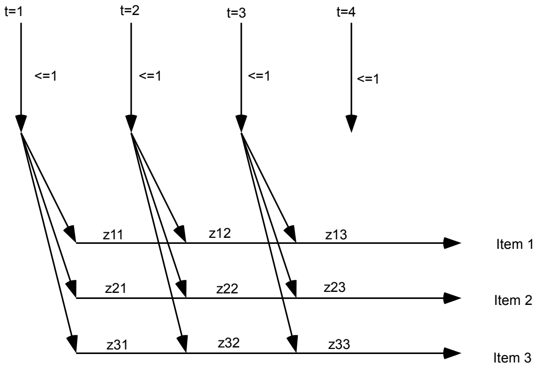
$$s_t^i \geq C^i \sum_{u=1}^{t} y_u^i - d_{1t}^i$$

and set $z_t^i = \sum_{u=1}^{t} y_u^i$

Reformulation

$$
\begin{aligned}
s_t^i/C^i &\geq z_t^i - d_{1t}^i/C^i \ \forall i,t \\
0 \leq z_t^i &- z_{t-1}^i \leq 1 \\
\sum_i (z_t^i - z_{t-1}^i) &\leq 1 \ \forall i,t
\end{aligned}
$$

This is a set of disjoint two variable MIPs plus network constraints.
Conclusion: Simple MIR inequalities give the convex hull for this multi-item problem.



t=1
t=2
t=3
t=4
<=1
<=1
<=1
<=1
z11
z12
z13
z21
z22
z23
z31
z32
z33
Item 1
Item 2
Item 3

Mixing Sets

The mixing set X^M consists of

$$s + z_l \geq b_l \text{ for } l = 1, \ldots, n$$
$$s \in \mathbb{R}^1,\ z \in \mathbb{Z}^n.$$

Let $f_l = b_l - \lfloor b_l \rfloor$ for all l.
A tight extended formulation for $\mathrm{conv}(X^M)$ is:

$$s = \sum_{i=1}^n f_i \delta_i + \mu$$
$$z_t + \mu + \sum_{\{i: f_i \geq f_t\}} \delta_i \geq \lfloor b_t \rfloor + 1 \text{ for } t = 1, \ldots, n$$
$$\sum_{i=0}^n \delta_i = 1$$
$$\delta \in \mathbb{R}_+^{n+1}, \mu \in \mathbb{R}^1, z \in \mathbb{R}^n.$$

A Network Dual Extended Formulation

Suppose wlog $1 = f_0 > f_1 \geq f_2 \cdots \geq f_n$. Let $\mu_0 = \mu$ and $\mu_t = \mu + \sum_{i:f_i \geq f_t} \delta_i$ giving:

$$
\begin{aligned}
s &= \textstyle\sum_{i=0}^{n} (f_i - f_{i+1})\mu_i \\
z_t + \mu_t &\geq \lfloor b_t \rfloor + 1 \text{ for } t = 1, \ldots, n \\
-\mu_{j-1} + \mu_j &\geq 0 \text{ for } j = 1, \ldots, n \\
\mu_0 - \mu_n &\geq -1 \\
\mu \in \mathbb{R}^{n+1}&, z \in \mathbb{R}^n.
\end{aligned}
$$

This is the transpose of a pure network matrix (*dual network*) matrix, and the extreme points are obviously integer.

Projection gives Mixing Inequalities

$$s = \sum_{i=0}^{n}(f_i - f_{i+1})\mu_i$$

$$\mu_t \geq \lceil b_t \rceil - z_t \text{ for } t = 1, \ldots, n$$

$$\mu_0 \geq \lceil b_n \rceil - 1 - z_n$$

Mixing Inequality

$$s \geq (1 - f_1)(\lceil b_n \rceil - 1 - z_n) + \sum_{i=1}^{n}(f_i - f_{i+1})(\lceil b_t \rceil - z_t)$$

Example of Mixing Inequality

Consider the set

$$X = \{(s, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^3 : s + y_1 \geq 0.7, s + y_2 \geq 2.6, s + y_3 \geq 1.4\}.$$

Mixing Inequality

$$s \geq (1-0.7)(1-y_3)+(0.7-0.6)(1-y_1)+(0.6-0.4)(3-y_2)+0.4(2-y_3).$$

Additional Constraints for Mixing

All faces are of the form: Some Mixing Inequality is tight, and $\alpha_{ij} \leq z_i - z_j \geq \beta_{ij}$ with $\alpha_{ij}, \alpha_{ij} \in \mathbb{Z}$.

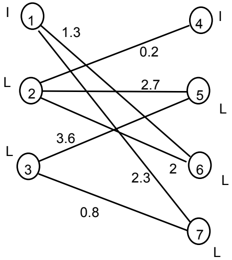
When is convex hull of $X^M \cap Q$ given by mixing inequalities?

When $Q = \{z \in \mathbb{R}^n : Dx \leq d\}$ with D a network dual matrix and d integer.

Generalization: Bipartite Edge Covering

Given $G = (V_1, V_2, E)$, consider the set

$$x_i + x_j \geq b_{ij} \ \forall (i,j) \in E$$
$$x_i \in \mathbb{Z}^1 \ i \in I, x_i \in \mathbb{R}^1 \ i \in L = V \setminus I$$



Network Dual MIPs

Given $D = (V, A)$, consider the set X^{ND}:

$$x_i - x_j \geq b_{ij} \; \forall (i, j) \in A$$
$$x_i \in \mathbb{Z}^1 \; i \in I, x_i \in \mathbb{R}^1 \; i \in L = V \setminus I$$

There exists a tight extended formulation of the form

$$\mu_{ijk} - \mu_{ijl(k)} \geq \beta_{ijk} \; \forall (i, j) \in A, k \in 1..Q$$
$$\mu_{ijk} \in \mathbb{R}^1 \; \forall \; i, j, k$$

with $\beta_{ijk} \in \mathbb{Z}$.
Its size depends on Q, the number of different fractional values the continuous variables take in the extreme points of $\mathrm{conv}(X)$.
Let $D_L = (L, A_L)$ be the digraph induced by the nodes corresponding to continuous variables.
Q is polynomial in size if D_L is a tree.

Open Questions

- Complexity when D_L is a bi-directed path.
- Membership Problem: Given $x^* \in \mathbb{R}^{|V|}$, decide whether $x^* \in \text{conv}(X^{ND})$.
- Every facet is induced by a tree in D_L. This would imply that Membership is in $co - NP$.

THANK YOU and ANY QUESTIONS

THEN over to JACK