Even pairs in Berge graphs

Journées POC, Paris, April 2009

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- A **hole** in a graph is an induced cycle of length at least 4, an **antihole** is the complement of a hole
- A graph is **Berge** if it contains no odd hole and odd antihole
- **Th [Chudnovsky, Robertson, Seymour and Thomas, 2002]:** a graph is perfect if and only if it is Berge.

Even pairs

An **even pair** is a pair of vertices of a graph such that all induced paths linking them are of even length

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- **Theorem [Fonlupt and Uhry, 1982]:** contracting an even pair of a graph preserves its chromatic number and the size of a largest clique
- **Theorem [Meyniel, 1987]:** a minimally imperfect graph contains no even pair

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  - deciding whether a graph contains an even-pair is CoNP-complete.

Polynomiality for Berge graphs

Follows easily in $O(n^9)$ from the Berge recognition algorithm
Chudnovsky, Cornuéjols, Liu, Seymour and Vušković, 2002

Proving that a class of graphs contains an even pair

Fist idea: start from an **induced** P_3

Result obtained:

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no interesting result...

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Result obtained:

- a **Meyniel graph** is a graph such that all odd cycles of length at least 5 admit at least 2 chords
- a Meyniel graph either is a clique or admits an even pair
- new proof of "all Meyniel graphs are perfect"
- Meyniel 1987

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Third idea: find a better vertex

Result obtained:

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Third idea: find a better vertex

Result obtained:

- if a graph contains no prism, no square and no odd hole then it is a clique or it admits an even pair
- all such graphs are perfect
- Linhares Sales and Maffray, 2002

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Fourth idea:

- consider a set of vertices as a vertex
- use the Roussel and Rubio Lemma

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Result obtained:

- an **Artemis graph** is a graph with no odd hole, no prism and no antihole of length at 5
- an artemis graph is a clique or admits an even pair, Maffray and Trotignon, 2002
- all artemis graphs are perfect
- coloring artemis graphs in time $O(n^2m)$ with Lévêque and Reed, 2004

First: how it was proved

Every Berge graph is **basic** or admits a **decomposition**

- take a Berge graph G.
- Suppose that G contains a well chosen induced subgraph H that easily satifies the Theorem.
  So H is "good": basic or has a decomposition.
- Prove that the rest of G must attach to H in a way that keeps "being good"
- From here on G can be assumed H-free.
- Go back to the first step with another good graph H.

The twelve classes

- About a **dozen** of steps of the decomposition process were needed by Chudnovsky, Robertson, Seymour and Thomas.
- $\mathcal{F}_0$: class of all Berge graphs
- $\mathcal{F}_1$: class of graphs from $\mathcal{F}_0$ where some kind of line-graph of a 3-connected graph is forbidden
- $\vdots$
- $\mathcal{F}_{11}$: class of all graphs from $\mathcal{F}_{10}$ with no antihole of length at least 6

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- Interestingly, all the famous "even pair" killers play the role of H at some step.
- Question: is there some $0 \leq i \leq 11$ such that all graphs in $\mathcal{F}_i$ are either a clique or admit an even pair?
- answer: Yes, $\mathcal{F}_{11}$ is included in Artemis
- something better: No

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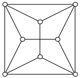
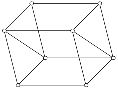
no  and no .

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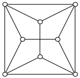
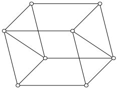
  no  and no .

- Decomposition of bipartisan graphs (Chudnovsky, Robertson, Seymour and Thomas):
  - Basics: bipartite graphs and their complement.
  - Operation: even skew partition.

The Chudnovsky and Seymour Theorem

- **Th [Chudnovsky and Seymour, 2007]:** every graph G in $\mathcal{F}_7$ has an even pair, or a dominant pair or a star cutset, or is a clique.
- $\mathcal{F}_7$: all graphs in $\mathcal{F}_6$ that contain no odd wheels.

Shorten the proof again ?

Maffray conjecture?

Generalizing even pairs?

- a pair of vertices is P_4-free if no path of length 3 link them
- a graph is P_4-contractile if it can be shrunk to a clique by a sequence of contraction of P_4-free pair
- **Conjecture [Lévêque, 2008]:** if a graph contains no odd hole and no antihole on at least 6 vertices then it is P_4-contractile

What about perfect graphs with no even pairs?

- Researchers including Chudnovsky, Seymour and Thomas conjecture that Berge graphs with no even pair can be fully constructed from basic graphs by few simple operations.
  Operations include: clique cutset, homogeneous set, 2-join ...
- This approach might lead to a combinatorial coloring algorithm for all perfect graphs.