Semidefinite Optimization - why bother ?

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Integer problems and Matrices

Optimizing over $x \in \{0,1\}^n$ can be transformed into optimizing over

$$X := xx^T$$

Since $\text{diag}(X) = x$, main diag of X takes the role of x.
Therefore any constraints on x translate to constraints on main diagonal of X.
Moreover, quadratic constraints in x translate into linear constraints in X.
But: Number of variables is squared.

Semidefinite optimization: Require X to be semidefinite.

Cliques and Lovasz theta function

$G = (V, E) \ldots$ Graph on n vertices.

$$\omega(G) = \max \sum_i x_i \text{ such that } x_i x_j = 0 \; ij \notin E, \; x_i \in \{0, 1\}$$

because feasible x must be characteristic vector of some clique.

Linearization trick: Consider $X = \frac{1}{x^T x} x x^T$.

X satisfies:

$$X \succeq 0, \;\; \text{tr}(X) = 1, \; x_{ij} = 0 \forall ij \notin E, \; rank(X) = 1$$

Note also: $e^T x = x^T x$, so $e^T x = \langle J, X \rangle$. Here $J = ee^T$.

Cliques and theta function (2)

Exercise: Show that

$$\omega = \max\{\langle J, X\rangle : X \succeq 0,\ \text{tr}(X) = 1,\ \ x_{ij} = 0\,(ij) \notin E, rk(X) = 1\}$$

The difficulty is hidden in the rank constraint.

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The difficulty is hidden in the rank constraint.
Lovasz (1979): relax the (diffcult) rank constraint
This gives semidefinite relaxation (= theta function).

$$\vartheta(G) := \max\{\langle J, X\rangle : X \succeq 0,\ \mathrm{tr}(X) = 1,\ \ x_{ij} = 0\ (ij) \notin E\}$$

This is a semidefinite program in the matrix variable X (of size n).

Copositive Connection

Schrijver (1979) improvement: include $X \geq 0$
In this case we can add up the constraints $x_{ij} = 0$ and get

$$\vartheta'(G) = \max\{\langle J, X\rangle : \langle A, X\rangle = 0, \text{tr}(X) = 1, X \geq 0, X \succeq 0\}.$$

($A \ldots$ adjacency of complement graph). We have
$\vartheta(G) \geq \vartheta'(G) \geq \omega(G)$.

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($A \ldots$ adjacency of complement graph). We have
$\vartheta(G) \geq \vartheta'(G) \geq \omega(G)$.
Replacing the cone $X \geq 0, X \succeq 0$ by $X \in COP$ gives copositive relaxation.

$$COP := \{X = VV^T : V \geq 0\},$$ completely positive matrices

$$CP := \{Y : a^T Y a \geq 0 \ \forall a \geq 0\}$$ copositive matrices

These cones are dual to each other.

Copositive Connection

Let A be adjacency matrix of graph, J be all ones matrix.
Theorem (DeKlerk and Pasechnik (SIOPT 2002))

$$\alpha(G) = \max\{\langle J, X\rangle : \langle A+I, X\rangle = 1, \quad X \in COP\}$$

$$= \min\{y : y(A+I) - J \in CP\}.$$

This is a copositive program with only one equation (in the primal problem).
This is a simple consequence of the Motzkin-Strauss Theorem.

Proof (1)

$$\frac{1}{\alpha(G)} = \min\{x^T(A+I)x : x \in \Delta\}$$ (Motzkin-Strauss Theorem)

$\Delta = \{x : \ \sum_i x_i = 1, \ x \geq 0\}$ is standard simplex. We get

$$0 = \min\{x^T(A + I - \frac{ee^T}{\alpha})x : \ x \in \Delta\}$$

$$= \min\{x^T(\alpha(A + I) - J)x : \ x \geq 0\}.$$

This shows that $\alpha(A + I) - J$ is copositive. Therefore

$$\inf\{y : \ y(A + I) - J \in CP\} \leq \alpha.$$

Proof (2)

Weak duality of copositive program gives:

$$\sup\{\langle J, X\rangle : \ \langle A+I, X\rangle = 1, \ X \in COP\} \leq$$

$$\leq \inf\{y : \ y(A+I) - J \in CP\} \leq \alpha.$$

Proof (2)

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Now let ξ be incidence vector of a stable set of size α. The matrix $\frac{1}{\alpha}\xi\xi^T$ is feasible for the first problem. Therefore

$$\alpha \leq \sup\{\ldots\} \leq \inf\{\ldots\} \leq \alpha.$$

This shows that equality holds throughout and sup and inf are attained.
The recent proof of this result by DeKlerk and Pasechnik does not make explicit use of the Motzkin Strauss Theorem.

Graph Coloring

Let $\mathcal{S}$ be the collection of stable sets in G. If $S \in \mathcal{S}$, we denote by x_S the characteristic vector of S.
A $k-$coloring of G is partition of $V(G)$ into k stable sets (=color classes).
Chromatic number $\chi(G)$:

$$\chi(G) = \min \sum \lambda_S \text{ such that } \sum_{S \in \mathcal{S}} \lambda_S x_S = e, \lambda_S \in \{0,1\}$$

Fractional chromatic number $\chi_f(G)$:

$$\chi(G) = \min \sum \lambda_S \text{ such that } \sum_{S \in \mathcal{S}} \lambda_S x_S = e, \lambda_S \geq 0.$$

(Integer) LP with exponential number of variables λ_S.

Coloring Matrices

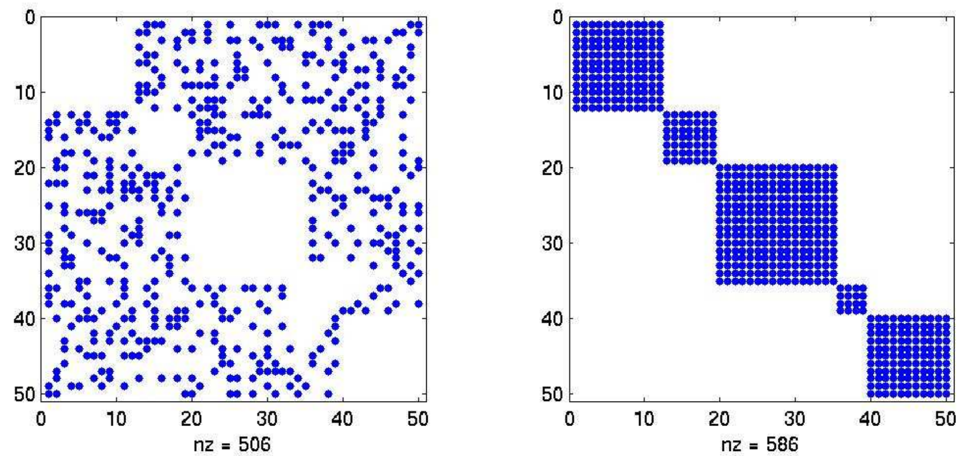
Suppose $\sum_S \lambda_S x_S = e$ and $\lambda_S \geq 0$.
Consider $X = \sum_{S \in \mathcal{S}} \lambda_S x_S x_S^T$.

Properties of X:

- $x_{ij} \in \{0, 1\}$ if $\lambda_S \in \{0, 1\}$,
- $x_{ij} = 0$ if $ij \in E(G)$,
- $diag(X) = e$ (because x_S is 0-1 vector)
- $M = \begin{pmatrix} \sum_S \lambda_S & e^T \\ e & X \end{pmatrix} \succeq 0,$

because $M = \sum_S \lambda_S \begin{pmatrix} 1 \\ x_S \end{pmatrix} \begin{pmatrix} 1 \\ x_S \end{pmatrix}^T \succeq 0.$

Coloring Matrices



Adjacency matrix A of a graph (left), associated Coloring Matrix (right). The graph can be colored with 5 colors.

Coloring as integer SDP

Exercise: Show that

$$\chi = \min\{\alpha : \begin{pmatrix} \alpha & e^T \\ e & X \end{pmatrix} \succeq 0, x_{ii} = 1, x_{ij} = 0 \; ij \in E, x_{ij} \in \{0,1\}\}$$

Coloring as integer SDP

Exercise: Show that

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Leaving out integrality of X, we get lower bound

$$\chi \geq \min\{\alpha : Y - J \succeq 0, diag(Y) = \alpha e, y_{ij} = 0 \; ij \in E\}$$

The SDP condition is a consequence of the Schur complement lemma with $Y = \alpha X$ and $J = ee^T$.

$$\begin{pmatrix} \alpha & e^T \\ e & X \end{pmatrix} \succ 0 \iff X - \frac{1}{\alpha} ee^T \succ 0.$$

Theta function again

$$\chi \geq \min\{\alpha : Y \succeq J,\ diag(Y) = \alpha e,\ y_{ij} = 0\ ij \in E\}$$

This SDP gives again the theta function (dual form).

Copositive refinement: Since $X = \sum_S \lambda_S x_S x_S^T$, we can further impose $Y \in COP$ (recall that $Y = \alpha X$):

$$\alpha^* := \min\{\alpha : Y \succeq J, Y \in COP, diag(Y) = \alpha e,\ y_{ij} = 0\ ij \in E\}$$

Taking feasible solution λ_S for $\chi_f(G)$, we note that $X = \sum_S \lambda_S x_S x_S^T$ is feasible, therefore

$$\alpha^* \leq \chi_f(G).$$

Dukanovic, Laurent, Gvozdenovic, R. (2007) $\alpha^* = \chi_f(G)$.

Some timings to compute theta function

The number of constraints depends on the edge set $|E|$. If m is small, then this SDP can be solved efficiently using interior point methods.

n	100	200	300	400
$\|E\|$	490	2050	4530	8000
time	2	52	470	2240
$\|E\|$	1240	5100	11250	20000
time	11	560	***	***

Times in seconds for computing $\vartheta(G)$ on random graphs with different densities(p = 0.1 and p=0.25).
In each iteration, a linear equation with $|E|$ variables has to be solved, so no hope if |E| > 10,000.

Some DIMACS graphs

graph	n	m	ϑ	ω
keller5	776	74.710	31.00	27
keller6	3361	1026.582	63.00	$\geq$59
san1000	1000	249.000	15.00	15
san400-07.3	400	23.940	22.00	22
brock400-1	400	20.077	39.70	27
brock800-1	800	112.095	42.22	23
p-hat500-1	500	93.181	13.07	9
p-hat1000-3	1000	127.754	84.80	$\geq$68
p-hat1500-3	1500	227.006	115.44	$\geq$94

Computations using boundary point method (Malick, Povh, Wiegele, R.(2007)). The theta number for the bigger instances has not been computed before .