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Lovász and Lehman Theorems on Clutters a common generalization

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Some definitions

$V=\{1,2,\ldots, n\}$

A set $\mathcal{A}$ of subsets of V is a ***clutter*** if

$$\nexists A_i, A_j \text{ in } \mathcal{A} \text{ such that } A_i \subset A_j$$

We associate to $\mathcal{A}$

a **matrix** A of size m x n : the rows are the characteristic vectors of the elements of $\mathcal{A}$

the **antiblocking polyhedron** $P_{\leq}(\mathcal{A}) = \{x \in R^n;\ Ax \leq 1 \text{ and } x \geq 0\}$

the **antiblocker** $\mathbf{b_{\leq}(\mathcal{A})} = \{B;\ B \subseteq V$ maximal such that $|B \cap A| \leq 1\ \forall A \in \mathcal{A}\}$: a clutter on V

A particular case

G =(V,E)

Clutter $\mathcal{A}(G)$ ={maximal cliques of G}

$\mathbf{b_{\leq}(\mathcal{A}(G))}$ = {B; B⊆V maximal such that |B∩K|≤1 ∀K ∈ $\mathcal{A}$}

= {maximal stable set of G}

<u>Theorem</u> (Lovász 1972) :

$P_{\leq}(\mathcal{A}(G))$ = {x∈R^n; Ax≤1and x≥0}= $b_{\leq}(\mathcal{A})$ iff G is perfect.

So if G is minimal imperfect then $P_{\leq}(\mathcal{A}(G))$ has some non integer vertex, but $P_{\leq}(\mathcal{A}(G'))$ any proper induced subgraph G' of G.

Lovász Theorem and Padberg Corollaries

Theorem : Let G=(V,E) be a minimal imperfect graph,

ω =maximum size of clique

α =maximum size of a stable set

then G

- has n= $\alpha\omega$ +1 vertices,
- contains exactly n ω-cliques $K_1, \ldots, K_n$ and n α-stable sets $S_1, \ldots, S_n$ and $K_i \cap S_j=1$ if $i \neq j$ and $K_i \cap S_i=0$,
- every vertex v of G belongs to exactly ω ω-cliques $K_{i1}, K_{i2}, \ldots, K_{i\omega}$,

  α α-stable sets $S_{j1}, S_{j2}, \ldots, S_{j\alpha}$

  and $S_{i1}, S_{i2}, \ldots, S_{i\omega}$ is a partition of V\v

  $K_{j1}, K_{j2}, \ldots, K_{j\alpha}$ is a partition of V\v

Lovász Theorem and Padberg Corollaries another formulation

Theorem : Let

$\mathcal{A}_G$ ={maximal cliques of a minimal imperfect graph G}

then

- $P_{\leq}(\mathcal{A}_G)$ is non integer, $(1/\omega, \ldots, 1/\omega)$ is its unique fractional vertex, and $P_{\leq}(\mathcal{A}_{G'})$ is integer for every proper induced subgraph G' of G,

and there exists

- X nxn matrix, rows =char. vectors of elements of $\mathcal{A}_G$
- Y nxn matrix columns=char. vectors of elements of $b_{\leq}(\mathcal{A}_G)$

such that X and Y are uniform and XY=YX=J-I

J = nxn all one matrix, I= nxn identity matrix, uniform = same number of 1 in each row and column, ω=max size of a clique, α= max size of a stable set.

Some other definitions

Given a clutter $\mathcal{A}$ on V

the matrix A of size m x n : rows = the characteristic vectors of the elements of $\mathcal{A}$

The **blocking polyhedron** $P_{\geq}(\mathcal{A}) = \{x \in R^n; Ax \geq 1 \text{ and } x \geq 0\}$

the **blocker**

$\mathbf{b_{\geq}(\mathcal{A})} = \{B; B \subseteq V \text{ maximal such that } |B \cap A| \geq 1 \ \forall A \in \mathcal{A}\}$: a clutter

Theorem (Edmonds-Fulkerson1970) : $b_{\geq}(b_{\geq}(\mathcal{A}))$.

$\mathcal{A}$ is said to be **ideal** if $P_{\geq}(\mathcal{A}) = \mathbf{b_{\geq}(\mathcal{A})}$

More definitions

Let x in R^n, i in V, P a polyhedron in R^n

The **projection** of x parallel to the ith coordinate is

$\mathbf{x}^i = (x_1, \ldots, , x_{i-1}, x_{i+1}, \ldots, x_n)$

Deletion : $P \backslash i = \{x^i ; x \in P\}$

Contraction : $P/i = \{x^i ; x \in P \text{ and } x_i = 0\}$

$\mathcal{A}$ is **minimally non ideal** if

$P_{\geq}(\mathcal{A}) = \{x \in R^n ; Ax \geq 1\}$ has at least one non-integer vertex but $\forall i \in V$ all vertices of $P \backslash i$ and P/i are integer

Some minimally non-ideal clutters

The degenerative projective plane clutter $\mathcal{F}_n (n \geq 3)$:

$\mathcal{F}_n = \{1, 2, \ldots, n-1\}, \{1,n\}, \{2,n\}, \ldots \{n-1,n\}\}$

$P_{\geq}(\mathcal{F}_n)$ has the fractional vertex

$(1/n-1, 1/n-1, \ldots, n-2/n-1)$

Lehman Theorem

Theorem (Lehman 1990)

Let $\mathcal{A}$ be a **minimally non ideal** clutter,

either $\mathcal{A} = \mathcal{F}_n$

or there exists

- **nxn** matrix X, rows =char. vectors of elements of $\mathcal{A}$
- **nxn** matrix Y columns=char. vectors of elements of $\mathcal{B}_{\geq}(\mathcal{A})$

such that X and Y are uniform and

$$XY=YX=J+(\mu-1)I \text{ for some } \mu\geq 2$$

Our theorem

Let $\mathcal{A}_{\leq}$ and $\mathcal{A}_{\geq}$ two clutters

and P:= $P_{\leq}(\mathcal{A}_{\leq}) \cap P_{\geq}(\mathcal{A}_{\geq})$ be minimally non integer, then

either $\mathcal{A}_{\leq}=\emptyset$, $\mathcal{A} = \mathcal{F}_n$ and w=(1/n-1, 1/n-1, …, n-2/n-1) isa unique fractionnal vertex of P

Or one or both of the following hold :

$\mathcal{A}_{\leq}$ is as in the case of Lovasz theorem and $(1/r_{\leq}, 1/r_{\leq}, \ldots, 1/r_{\leq})$ is a vertex of P

$\mathcal{A}_{\geq}$ is as in the case of Lehman theorem and $(1/r_{\geq}, 1/r_{\geq}, \ldots, 1/r_{\geq})$

One key element of the proofs

The commutativity Lemma :

If X and Y are two nxn (0,1) matrices

and XY are such that all non diagonal elements are equal to 1 and the diagonal elements are either all equal to 0 or all >1 then

X uniform ⇒ Y is uniform too, all diagonal elements are equal and XY=YX

Happy birthday Jack