

# Lovász and Lehman Theorems on Clutters a common generalization

**Grigor GASPARYAN, Myriam PREISSMANN, András SEBŐ**

Laboratoire G-SCOP  
46, av Félix Viallet  
38031 Grenoble Cedex  
[www.g-scop.inpg.fr](http://www.g-scop.inpg.fr)

$$V = \{1, 2, \dots, n\}$$

A set  $\mathcal{A}$  of subsets of  $V$  is a **clutter** if

$$\nexists A_i, A_j \text{ in } \mathcal{A} \text{ such that } A_i \subset A_j$$

We associate to  $\mathcal{A}$

a **matrix**  $\mathbf{A}$  of size  $m \times n$  : the rows are the characteristic vectors of the elements of  $\mathcal{A}$

the **antiblocking polyhedron**  $P_{\leq}(\mathcal{A}) = \{x \in \mathbb{R}^n; \mathbf{A}x \leq 1 \text{ and } x \geq 0\}$

the **antiblocker**  $\mathbf{b}_{\leq}(\mathcal{A}) = \{B; B \subseteq V \text{ maximal such that } |B \cap A| \leq 1 \forall A \in \mathcal{A}\}$  : a clutter on  $V$

$$G = (V, E)$$

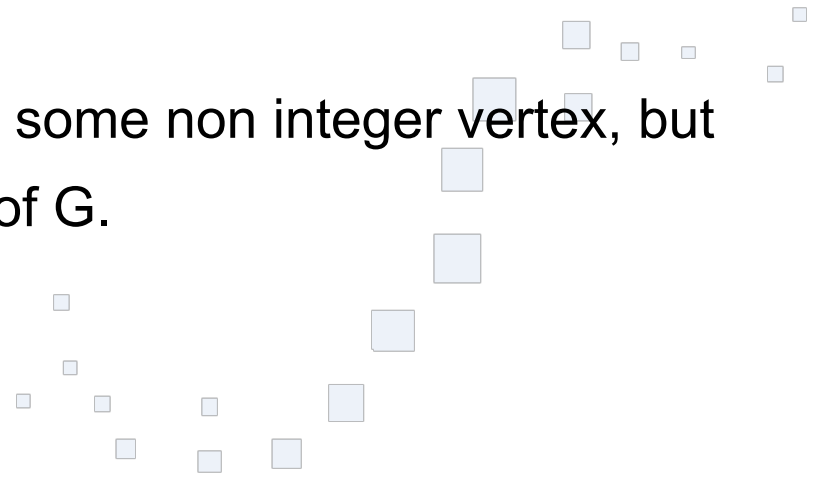
Clutter  $\mathcal{A}(G) = \{\text{maximal cliques of } G\}$

$$\begin{aligned} b_{\leq}(\mathcal{A}(G)) &= \{B; B \subseteq V \text{ maximal such that } |B \cap K| \leq 1 \ \forall K \in \mathcal{A}\} \\ &= \{\text{maximal stable set of } G\} \end{aligned}$$

Theorem (Lovász 1972) :

$$P_{\leq}(\mathcal{A}(G)) = \{x \in \mathbb{R}^n; Ax \leq 1 \text{ and } x \geq 0\} = b_{\leq}(\mathcal{A}) \text{ iff } G \text{ is perfect.}$$

So if  $G$  is minimal imperfect then  $P_{\leq}(\mathcal{A}(G))$  has some non integer vertex, but  $P_{\leq}(\mathcal{A}(G'))$  any proper induced subgraph  $G'$  of  $G$ .



Theorem : Let  $G=(V,E)$  be a minimal imperfect graph,

$\omega$  =maximum size of clique

$\alpha$  =maximum size of a stable set

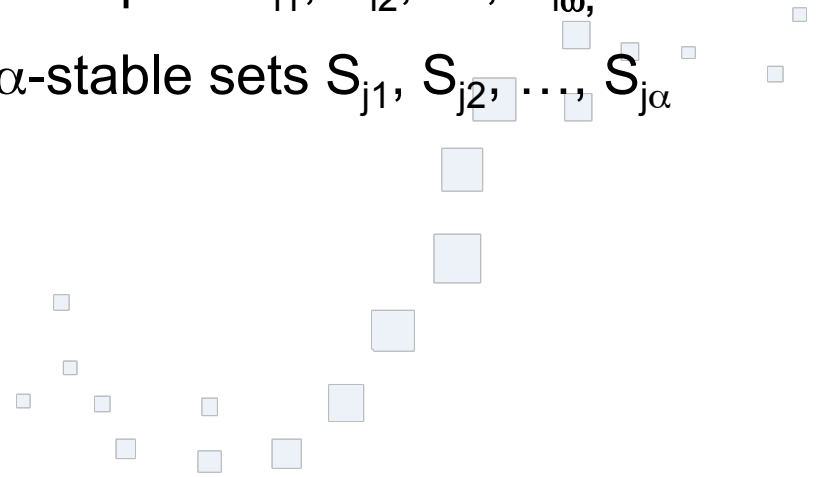
then  $G$

- has  $n= \alpha\omega +1$  vertices,
- contains exactly  $n$   $\omega$ -cliques  $K_1, \dots, K_n$  and  $n$   $\alpha$ -stable sets  $S_1, \dots, S_n$  and  $K_i \cap S_j = 1$  if  $i \neq j$  and  $K_i \cap S_i = 0$ ,
- every vertex  $v$  of  $G$  belongs to exactly  $\omega$   $\omega$ -cliques  $K_{i_1}, K_{i_2}, \dots, K_{i_\omega}$ ,

$\alpha$   $\alpha$ -stable sets  $S_{j_1}, S_{j_2}, \dots, S_{j_\alpha}$

and  $S_{i_1}, S_{i_2}, \dots, S_{i_\omega}$  is a partition of  $V \setminus v$

$K_{j_1}, K_{j_2}, \dots, K_{j_\alpha}$  is a partition of  $V \setminus v$



Theorem : Let

$\mathcal{A}_G = \{\text{maximal cliques of a minimal imperfect graph } G\}$

then

- $P_{\leq}(\mathcal{A}_G)$  is non integer,  $(1/\omega, \dots, 1/\omega)$  is its unique fractional vertex, and  $P_{\leq}(\mathcal{A}_{G'})$  is integer for every proper induced subgraph  $G'$  of  $G$ ,

and there exists

- $X$   $n \times n$  matrix, rows = char. vectors of elements of  $\mathcal{A}_G$
- $Y$   $n \times n$  matrix columns = char. vectors of elements of  $b_{\leq}(\mathcal{A}_G)$

such that  $X$  and  $Y$  are uniform and  $XY = YX = J - I$

$J = n \times n$  all one matrix,  $I = n \times n$  identity matrix, uniform = same number of 1 in each row and column,  $\omega = \text{max size of a clique}$ ,  $\alpha = \text{max size of a stable set}$ .

# Some other definitions

Given a clutter  $\mathcal{A}$  on  $V$

the matrix  $A$  of size  $m \times n$  : rows = the characteristic vectors of the elements of  $\mathcal{A}$

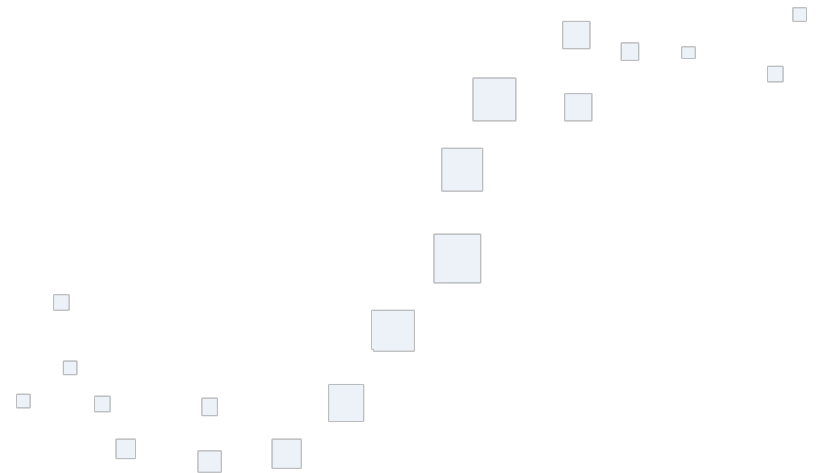
The **blocking polyhedron**  $P_{\geq}(\mathcal{A}) = \{x \in \mathbb{R}^n; Ax \geq 1 \text{ and } x \geq 0\}$

the **blocker**

$b_{\geq}(\mathcal{A}) = \{B; B \subseteq V \text{ maximal such that } |B \cap A| \geq 1 \ \forall A \in \mathcal{A}\}$  : a clutter

Theorem (Edmonds-Fulkerson 1970) :  $b_{\geq}(b_{\geq}(\mathcal{A}))$ .

$\mathcal{A}$  is said to be **ideal** if  $P_{\geq}(\mathcal{A}) = b_{\geq}(\mathcal{A})$



Let  $x$  in  $\mathbb{R}^n$ ,  $i$  in  $V$ ,  $P$  a polyhedron in  $\mathbb{R}^n$

The **projection** of  $x$  parallel to the  $i$ th coordinate is

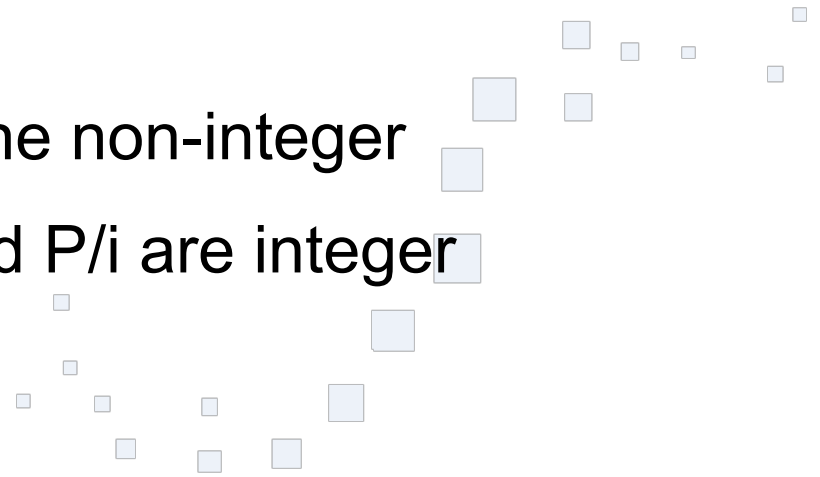
$$\mathbf{x}^i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

**Deletion** :  $P \setminus i = \{x^i; x \in P\}$

**Contraction** :  $P / i = \{x^i; x \in P \text{ and } x_i = 0\}$

$\mathcal{A}$  is **minimally non ideal** if

$P_{\geq}(\mathcal{A}) = \{x \in \mathbb{R}^n; Ax \geq 1\}$  has at least one non-integer vertex but  $\forall i \in V$  all vertices of  $P \setminus i$  and  $P / i$  are integer



The degenerative projective plane clutter  $\mathcal{F}_n (n \geq 3)$  :

$$\mathcal{F}_n = \{1, 2, \dots, n-1\}, \{1, n\}, \{2, n\}, \dots, \{n-1, n\}$$

$P_{\geq}(\mathcal{F}_n)$  has the fractional vertex

$$(1/n-1, 1/n-1, \dots, n-2/n-1)$$





Theorem (Lehman 1990)

Let  $\mathcal{A}$  be a **minimally non ideal** clutter,

either  $\mathcal{A} = \mathcal{F}_n$

or there exists

- **$n \times n$**  matrix  $X$ , rows = char. vectors of elements of  $\mathcal{A}$

- **$n \times n$**  matrix  $Y$  columns = char. vectors of elements of  $\mathcal{B}_{\geq}(\mathcal{A})$

such that  $X$  and  $Y$  are uniform and

$$XY = YX = J + (\mu - 1)I \text{ for some } \mu \geq 2$$



Let  $\mathcal{A}_{\leq}$  and  $\mathcal{A}_{\geq}$  two clutters

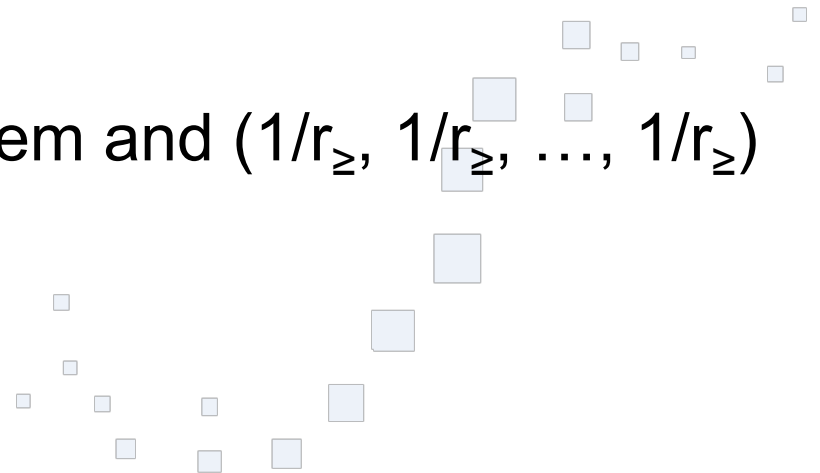
and  $P := P_{\leq}(\mathcal{A}_{\leq}) \cap P_{\geq}(\mathcal{A}_{\geq})$  be minimally non integer, then

either  $\mathcal{A}_{\leq} = \emptyset$ ,  $\mathcal{A} = \mathcal{F}_n$  and  $w = (1/n-1, 1/n-1, \dots, n-2/n-1)$  is a unique fractionnal vertex of  $P$

Or one or both of the following hold :

$\mathcal{A}_{\leq}$  is as in the case of Lovasz theorem and  $(1/r_{\leq}, 1/r_{\leq}, \dots, 1/r_{\leq})$  is a vertex of  $P$

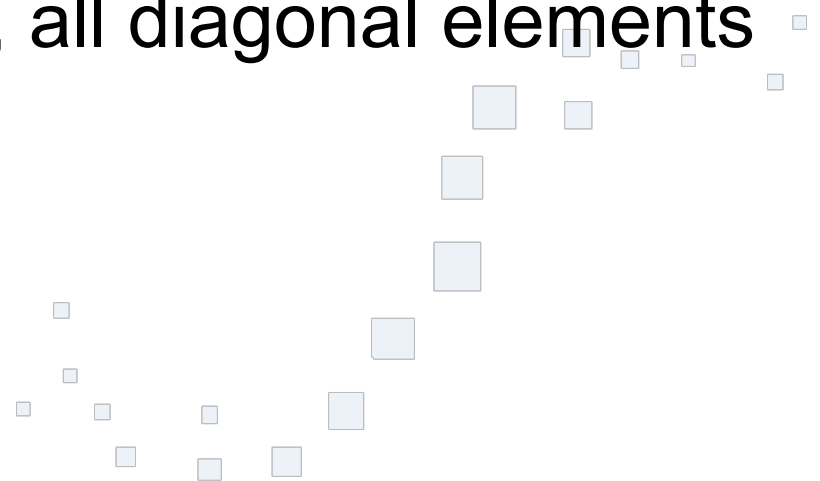
$\mathcal{A}_{\geq}$  is as in the case of Lehman theorem and  $(1/r_{\geq}, 1/r_{\geq}, \dots, 1/r_{\geq})$



The commutativity Lemma :

If  $X$  and  $Y$  are two  $n \times n$   $(0,1)$  matrices and  $XY$  are such that all non diagonal elements are equal to 1 and the diagonal elements are either all equal to 0 or all  $>1$  then

$X$  uniform  $\Rightarrow Y$  is uniform too, all diagonal elements are equal and  $XY=YX$



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*Happy birthday Jack*

