On the connectivity of the *k*-clique polyhedra

J. F. Maurras and R. Nedev

LIF, Univ. de la Méditerranée

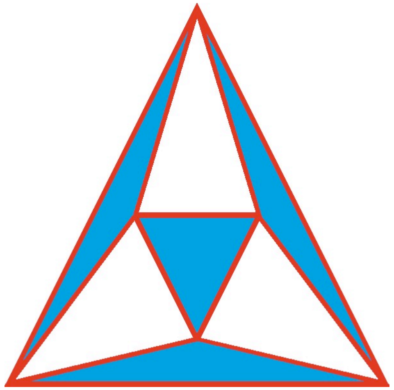
Introduction

- Let P_{nk} be the polyhedron of the edges incidence vectors X_k of the cliques with k vertices (k-cliques) of K_n, the complete graph with n vertices.

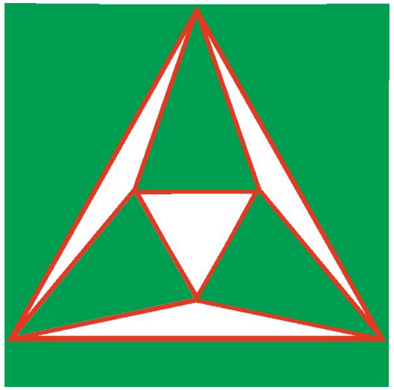
$$P_{nk} = conv(X_k)$$

- A polyhedron P is *h-neighbourly* if every subset W of h vertices is the set of vertices of a face of P.

Neighbourlicity



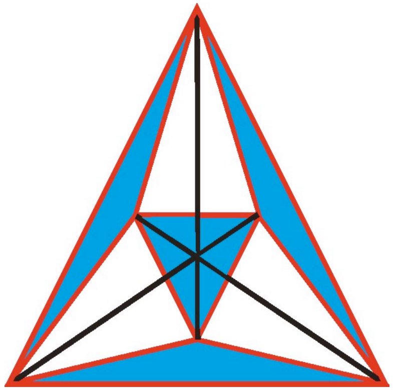
$$\sum_{e \in E(K_b)} \alpha_e = \beta$$



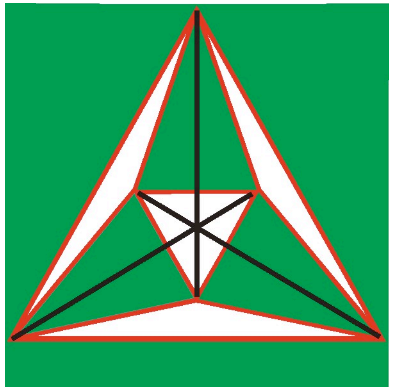
$$\sum_{e \in E(K_g)} \alpha_e < \beta$$

Contradiction: $\beta < \beta$

Neighbourlicity



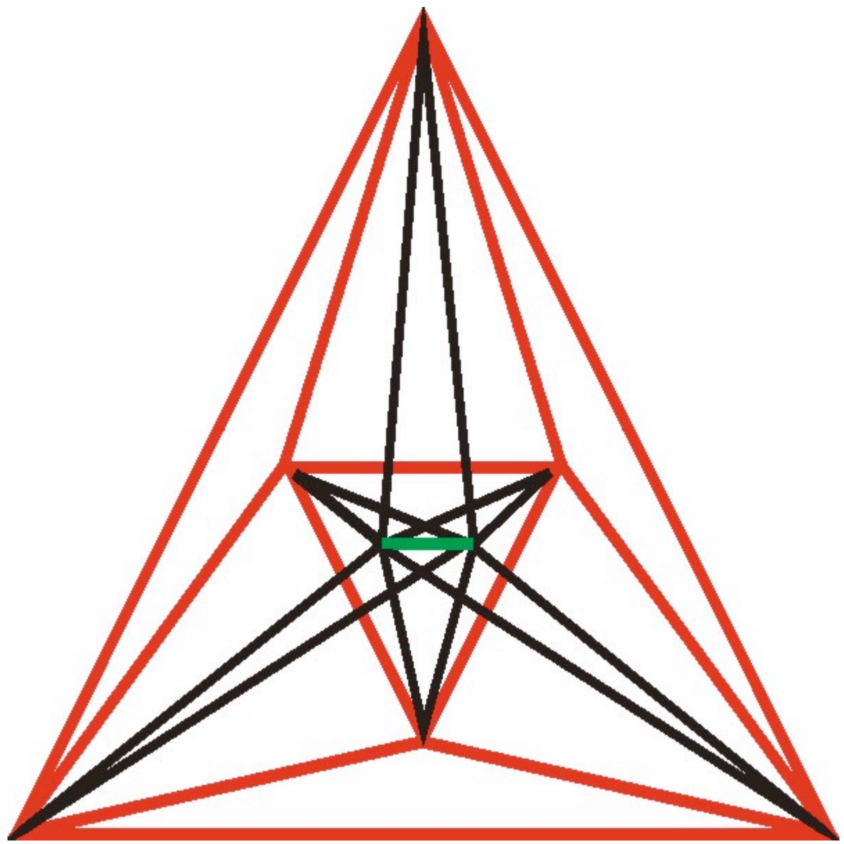
$$\sum \alpha_e + 2\sum \alpha_n = \beta$$
$e \in E(K_b)$, $n \in E(K_n)$



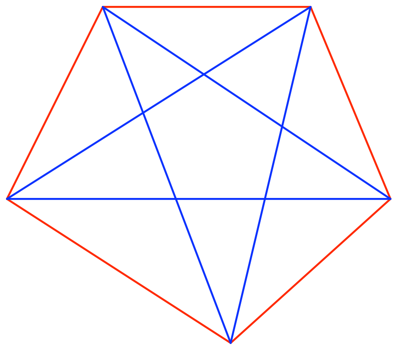
$$\sum \alpha_e + 2\sum \alpha_n < \beta$$
$e \in E(K_g)$, $n \in E(K_n)$

The same contradiction: $\beta < \beta$

Neighbourlicity



5-Cliques



$\gamma_1 \;: \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1$

$\gamma_2 \;: \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} = 1$

$i \neq 1,2 : \gamma_i \;: \Sigma e \in \gamma_i \; \alpha_e < 1$

Multipliers -1 for the two equations and 1/5 for the ten inequations yields 0 < 0. Thus the 5-cycle polyhedron is not 2-neighbourly!

Neighbourlicity

Suppose that P_{nk} is not 3-neighbourly, then there exists 3 k-cliques C_1, C_2, C_3, s.t. the system :

$$\forall i \in \{1,2,3\}, \quad \sum_{e\in E} \alpha_e x^i_e \geq 1,$$

$$\forall C \neq C_1, C_2, C_3, \sum_{e\in E} \alpha_e x^i_e < 1,$$

is impossible.

Thus there exists $\lambda_1, \lambda_2, \lambda_3 \leq 0$, not all zero, and $\lambda_C \geq 0$ st :

$$\forall e \in E, \; \lambda_1 x_e^{C_1} + \lambda_2 x_e^{C_2} + \lambda_3 x_e^{C_3} + \sum_{C\notin\{C_1,C_2,C_3\}} \lambda_e x_e^{C} = 0,$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \sum_{C\notin\{C_1,C_2,C_3\}} \lambda_e \leq 0.$$

Support condition

The last inequality implies that the support (i.e. the graph the edges of which have a non-zero coefficient) of the second set of cliques has to be included in the one of the first.

Obviously, in order that the left hand side of :

$$\forall e \in E,\ \lambda_1 x_e^{C_1} + \lambda_2 x_e^{C_2} + \lambda_3 x_e^{C_3} + \sum_{C \notin \{C_1, C_2, C_3\}} \lambda_e x_e^{C} = 0$$

can be ≥ 0, both supports have to be equal.

Neighbourlicity

1. Suppose w.l.o.g. that C_1 has a vertex $x \notin V(C_2) \cup V(C_3)$, its star can only be covered by C_1 . Thus P_{nk} is 'obviously' 2-neighbourly. To make zero the left part we need C_1 and analogously C_2 and C_3 $(C_2 \neq C_3)$.

2. Thus w.l.o.g. $V(C_1) \subset V(C_2) \cup V(C_3)$.

3. We denote $V(C_{1,2,3})$, (resp. $E(C_{1,2,3})$) the common vertices (resp. edges) of C_1, C_2, C_3. For $i,j \in \{1,2,3\}$, $V(C_{i,j})$, (resp. $E(C_{i,j})$) the common vertices (resp. edges) of C_i, C_j and $V(C_i)$, (resp. $E(C_i)$) the vertices (resp. edges) belonging only to C_i.

Neighbourlicity

2-1. E(C_i), E(C_1) for example, is the edge-set of the complete bipartite graph with vertex sets:

$V(C_{1,3})\backslash V(C_{1,2,3})$ and $V(C_{1,2})\backslash V(C_{1,2,3})$

- Consider the graph with values $\lambda_1 x^{C_1} + \lambda_2 x^{C_2} + \lambda_3 x^{C_3}$ assigned to the edges. Suppose that the same value can be obtain with other cliques. The edges of C_1 with value $\lambda_1 + \lambda_2 + \lambda_3$ are contained in *all* cliques. The edges with value λ_1 are contained only in the clique with $E(C_1)$.
- Thus the union of the these edges has a vertex set $V(C_1)$ and it forms the unique clique C_1.

Neighbourlicity

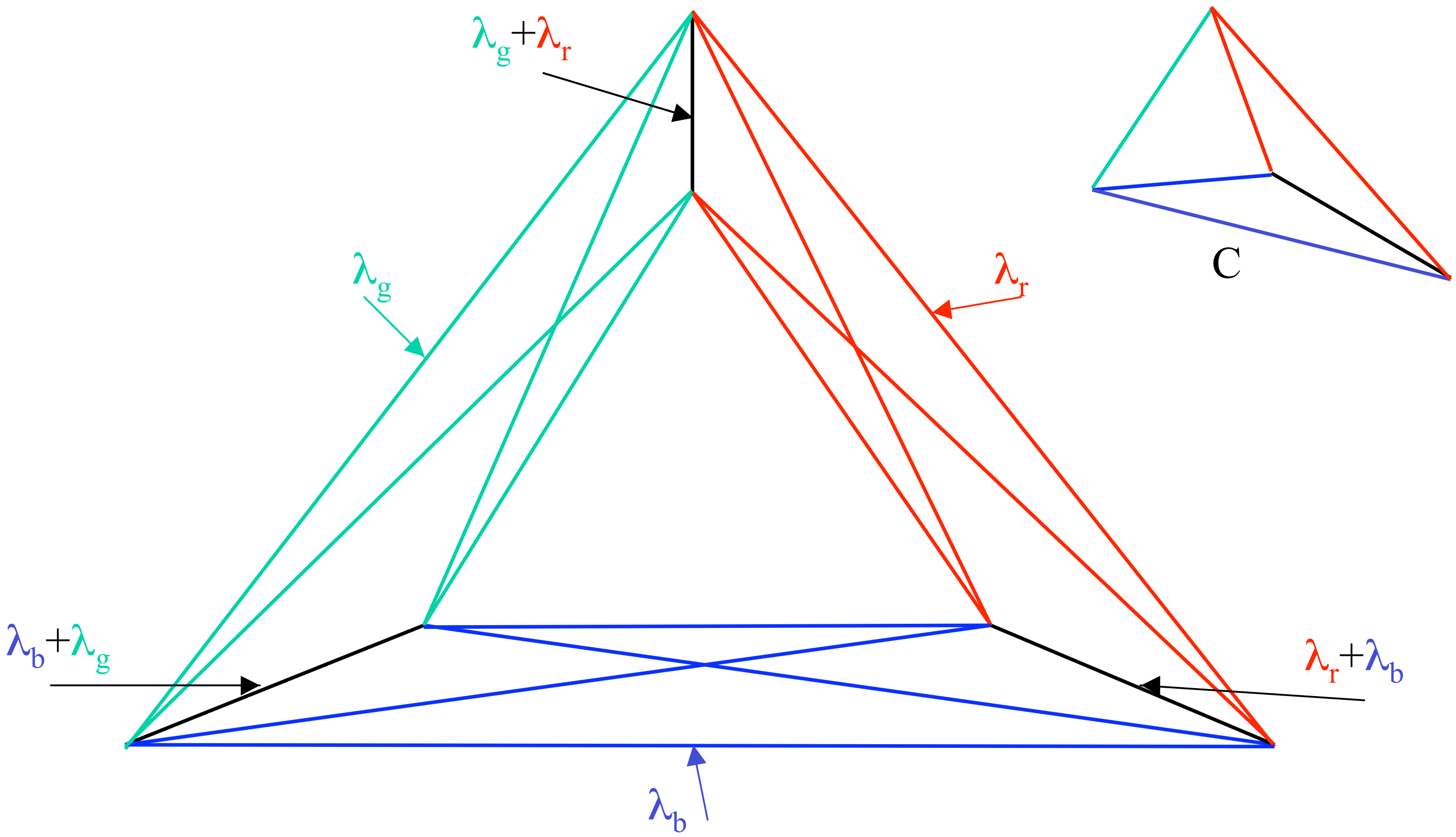
2-2. Suppose that a clique of $C \backslash \{C_1, C_2, C_3\}$, say K has some vertices in $V(C_{1,2}) \backslash V(C_{1,2,3})$, in $V(C_{2,3}) \backslash V(C_{1,2,3})$ and in $V(C_{1,3}) \backslash V(C_{1,2,3})$.

If $\lambda_K > 0$, *we can never have :*

$$\forall e \in E,\ \lambda_1 x_e^{C_1} + \lambda_2 x_e^{C_2} + \lambda_3 x_e^{C_3} + \sum_{C \notin \{C_1, C_2, C_3\}} \lambda_e x_e^{C} = 0.$$

Consequently a clique of $C \backslash \{C_1, C_2, C_3\}$ has vertices in, for instance,$V(C_{1,2}) \backslash V(C_{1,2,3})$ and in $V(C_{2,3}) \backslash V(C_{1,2,3})$, and thus is, in this case, is C_2…

Neighbourlicity



With the selected clique $C \notin \{C_b, C_g, C_r\}$, we will have definitely a deficiency of weight on the black edges.

Neighbourlicity

2-2. The previous proof supposed that either: $E(C_{1,2,3}) \neq \emptyset$ or $C_{1,2,3} \neq \emptyset$. Thus suppose $C_{1,2,3} = \{v\}$, and suppose that one clique that contains $E(C_1)$ does not contain v but $u \in (V(C_2) \cap V(C_3)) \setminus \{v\}$.

It follows that there is an edge from $E(C_{2,3})$ that contains u with value greater than λ_2 and λ_3, a contradiction.

Hypergraphs . Neighbourlicity.

- Let $K_n^r = (X,E)$ be the complete r-uniforme hypergraph with $|X| = n$ and $E = \{e \subset X, |e| = r\}$. As before, we will study the neighbourlicity of the convex hull P_{nk}^r of the k-cliques.

- We will search for the least number of cliques which share the same edges.

Hypergraphs . Neighbourlicity.

Consider a set of k-cliques indexed by $J \subset I$ (the set of all cliques) which do not form a face of $P_{nk}^{\ r}$, i.e.the system:

$$\forall j \in J, \quad \sum_{e \in E} \alpha_e x^j_e = \beta,$$

$$\forall i \in I \backslash J, \sum_{e \in E} \alpha_e x^i_e < \beta,$$

has no solution (α_E, β).

Hypergraphs. Neighbourlicity.

The previous system doesn't have a solution *iff* there are $\mu_i \leq 0$, not all zero and $\lambda_j \geq 0$, s.t. the system:

$$\forall e \in E, \quad \sum_{i \notin} \mu_i X^i_e + \sum_{j \in J} \lambda_j X^j_e = 0$$

$$\sum_{i \notin} \mu_i \beta + \sum_{j \in J} \lambda_j \beta \leq 0$$

has a solution.

This leads us to the following model which gives us the neighbourliclity of the polyhedron.

Hypergraphs. Neighbourlicity

Mixed boolean program:

$$\begin{cases} \min \sum_{i \in C_k} x_i, \\ \sum_{e \in K_i} x_i - \sum_{e \in K_i} y_i = 0, \forall e \in E, \\ x_i \leq \binom{n-r}{k-r} X_i, y_i \leq \binom{n-r}{k-r} Y_i, \forall i \in C_k, \\ X_i + Y_i \leq 1, \forall i \in C_k, \\ \sum_{v \ni K_i} x_i \geq 2, \forall v \in U \subset V, \mid U \mid = k+1, \\ x_i \geq 0, y_i \geq 0, x_i \in \mathbb{Z}, X_i, Y_i \in \{0,1\}. \end{cases}$$

The value of the solution of this program is the neighbourlicity of the polyhedron + 1.

Hypergraphs. Neighbourlicity

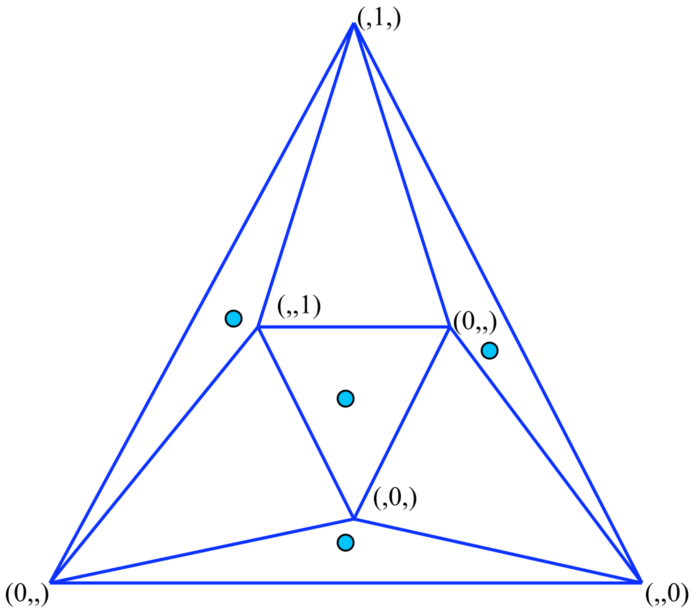
Exact values of neighbourlicity

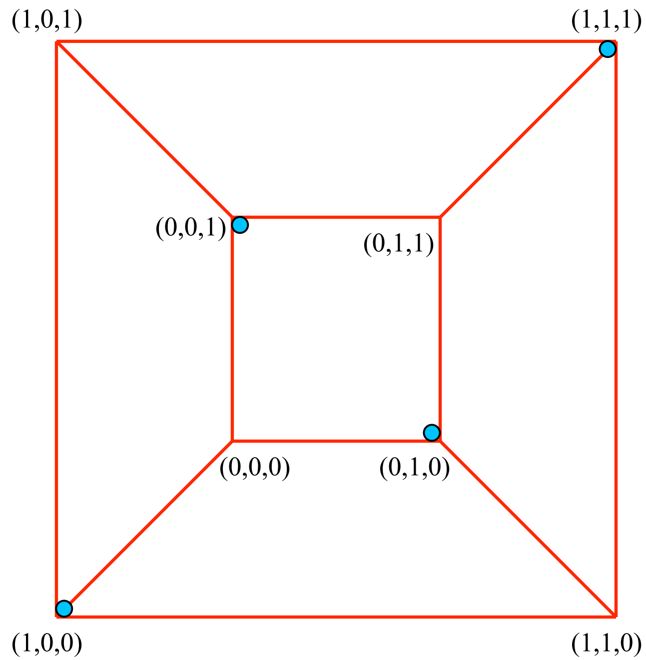
$n \geq 2(r+1)$	k	r	$neighbourlicity$
$\geq k+3$	≥ 3	2	3
8	4	3	7
9	4	3	7
9	5	3	7
10	5	3	7
10	6	3	7
10	5	4	15
11	6	4	15
12	6	5	31

Neighbourlicity. An upper bound

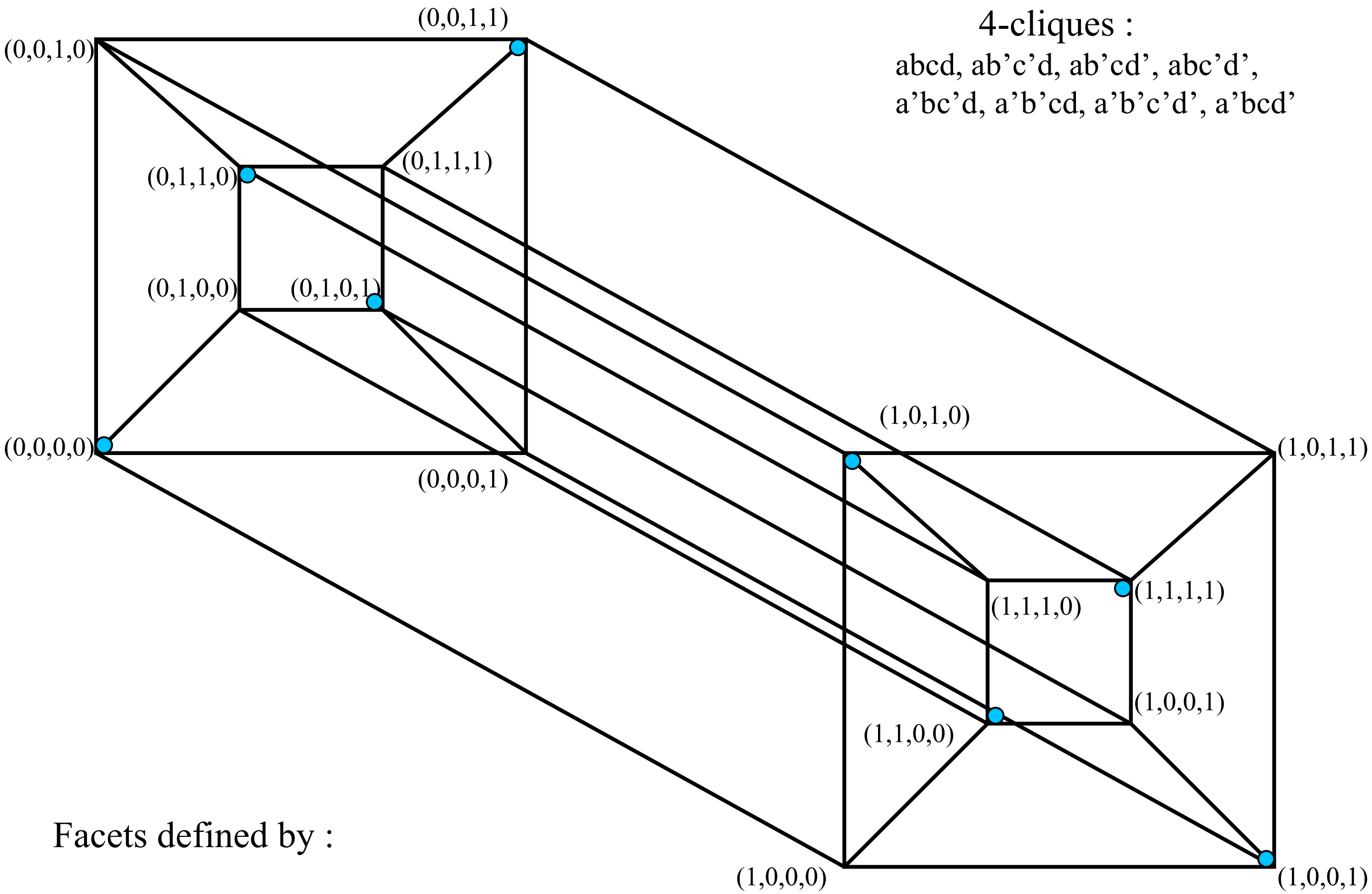
- Consider the subgraph $C_n(r)$ of K_n^r, s.t. $C_n(r)$ is the edge graph of the cross polytope with $(r+1)$ dimensions.
- There is a bijection between the *maximum cliques* of $C_n(r)$ and the *vertices* of the unit-hupercube with $(r+1)$ dimensions.
- An upper bound of neighbourlicity can be obtained from the maximum stable set of the unit hypercube with $(r+1)$ dimensions.

The octahedron and its dual





Hypercube of dimension 4



4-cliques :

abcd, ab'c'd, ab'cd', abc'd',
a'bc'd, a'b'cd, a'b'c'd', a'bcd'

Facets defined by :

a : $x \geq 0$, a' : $x \leq 1$, b : $y \geq 0$, b': $y \leq 1$,
c : $z \geq 0$, c' : $z \leq 1$, d : $t \geq 0$, d' : $t \leq 1$.

Hypergraphs. Unit-hypercube

- As before, we can generalize the upper bound of neighbourlicity by induction. The following argument can be used: If we assume that the *d*-dimensional hyper-cube contains a stable set with cardinality *M*, the *(d+1)*-dimensional hyper-cube contains a stable set of cardinality *2M*, as its edge-graph do not contain a cycle of odd lenght.

- Thus an upper bound of neighbourlicity for P_{nk}^{r} is $2^{r} - 1$.