Linear Programming in Oriented Matroids and the Active Bijection

Michel Las Vergnas
CNRS,Paris

joint work with

Emeric Gioan
CNRS,Montpellier

april 8, 2009

dedicated to Jack Edmonds
on the occasion of his 75th birthday

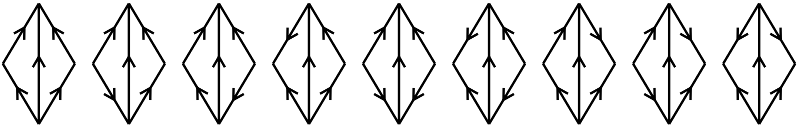
Two Classical Results

(1) The number of acyclic orientations of a graph G is equal to $|\chi(G;-1)|$, where $\chi(G;q)$ is the chromatic polynomial of G, i.e. the number of proper colorings of G in q colors.

This result has been proved for graphs by R. Stanley in 1973. It is also a corollary of a more general result for hyperplane arrangements due to R.O. Winder (1966).

Example

$\chi(G;q) = q(q-1)(q-2)^2$ $\qquad$ $2.3^2 = 18$ acyclic orientations



Two Classical Results

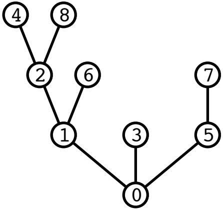
Let $p = a_1 a_2 \ldots a_n$ be a permutation of the integers $1, 2, \ldots, n$. We define a graph on $n+1$ vertices v_i $0 \leq i \leq n$ as follows. For $i = 1, 2, \ldots, n$, if $a_\ell < a_i$ for some integer $1 \leq \ell < i$, then we introduce the edge $v_j v_i$, where j is the greatest such integer, otherwise we introduce the edge $v_0 v_i$.

The graph $t(p)$ defined by the above edges is an increasing tree.[1]

(2) The mapping $p \mapsto t(p)$ is a bijection between n-permutations and increasing trees on $n+1$ vertices labeled by $0, 1, \ldots, n$.

This bijection has been introduced independently by W.H. Burge (1972), J. Françon (1976), X. Viennot (1976). It is now well-known and basic in enumerative combinatorics. See for instance, R.Stanley, *Enumerative combinatorics I* (1986), p. 25.

[1] A tree on vertices $0, 1, \ldots, n$ is *increasing* if labels increase along any path starting from 0.



$t(57316284)$

the bijection for $n = 3$

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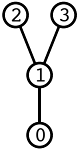

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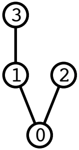

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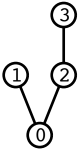

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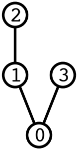

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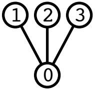

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Purpose of the Talk

At first sight, (1) and (2) seem unrelated. Our purpose in the talk is to present a bijection introduced by the authors in recent papers - called the **active bijection** - containing both (1) and (2) as corollaries, and thus unifying them.

The active bijection is closely related to linear programming. Unexpected links follow between (1) (2) and LP, and more generally, between the **Tutte polynomial** and LP.

A natural context for the active bijection is provided by **oriented matroids**. This generalization requires the theory of linear programming in oriented matroids, also called **pseudolinear programming**. For the convenience of the audience, in the first part of the talk, we will briefly recall the main features of pseudolinear programming.

Oriented Matroids

In this talk, oriented matroids are basically combinatorial abstractions of (signed) affine hyperplane arrangements.

Let $\mathcal{A}$ be a hyperplane arrangement in R^d. The oriented matroid $M(\mathcal{A})$ is defined by the collection of all sign-vectors with components $+, -, 0$ giving the positions of the points of R^d with respect to the hyperplanes in $\mathcal{A}$. These sign-vectors are the *covectors* of $M(\mathcal{A})$.

In general, an oriented matroid on a set E defined by its covectors is given by a collection of sign-vectors in $(+, -, 0)^E$ satisfying certains axioms. It turns out that these axioms amount to generalizing arrangements of (signed) real hyperplanes to arrangements of (signed) pseudohyperplanes (intuitively, waved hyperplanes, intersecting like real projective hyperplanes).

Oriented Matroids

We will use here the **hemispherical representation** of affine oriented matroids. The space is the completion of $\mathbf{R}^d$ homeomorphic to $\mathbf{B}^d$.

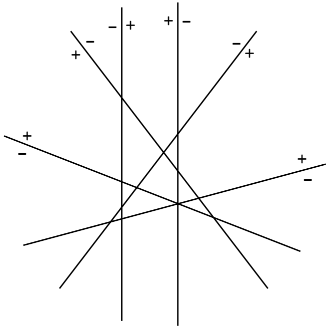
The boundary $p \approx \mathbf{S}^{d-1}$ of $\mathbf{B}^d$ is the **hyperplane at infinity**. Note that opposite points of p are not identified.

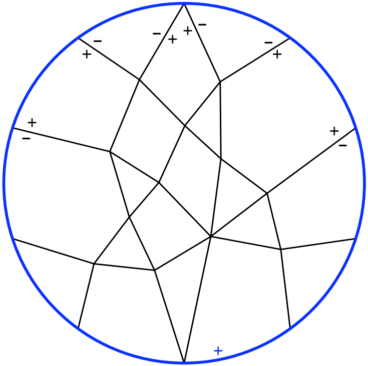
In a d-dimensional arrangement of pseudohyperplanes, two distinct pseudohyperplanes have a $(d-1)$-dimensional intersection. They are **parallel** if their intersection is contained in p.

A region is **bounded** if it has no vertex in p.

Our actual examples, all in dimension 2 (matroid rank 3), are **arrangements of pseudolines**.

Oriented Matroids





A real (signed) affine arrangement of hyperplanes in dimension 2, and its oriented matroid as an arrangement of (signed) pseudolines.

• Most oriented matroids cannot be represented by real hyperplane arrangements.

Linear Programming

A dimension-d **linear program** $\mathcal{P}$ is defined by an affine **hyperplane arrangement** $\mathcal{A}$ in $\mathbf{R}^d$ and a linear form called the **objective function**.

For simplicity, we will always suppose here that the **feasible region** of $\mathcal{P}$ is the fundamental region of $\mathcal{A}$, intersection of the closed non negative halfspaces defined by the hyperplanes.

A **solution** to $\mathcal{P}$ is a point of the feasible region R maximizing f over R.

- **If the feasible region R is non empty and bounded, then the linear program $\mathcal{P}$ has at least one solution.**

Pseudolinear Programming

A **pseudolinear program** $\mathcal{P} = (\mathcal{A}; p, f)$ is defined by an afffine oriented matroid on $\mathcal{A} \cup \{f\}$, with **hyperplane at infinity** $p \in \mathcal{A}$. The **objective function** $f \neq p$ may be in $\mathcal{A}$, or not.

The **feasible region** of $\mathcal{P}$ will always be the fundamental region of $\mathcal{A}$, i.e. the intersections of all closed non negative halfspaces defined by hyperplanes in $\mathcal{A}$.

The definition of maximizing the objective function over the feasible region is more elaborate than in the real case, as we cannot simply evaluate a linear form.

Consider an edge of the feasible region R not parallel to f. Its supporting pseudoline - a 1-dimensional intersection of hyperplanes of $\mathcal{A}$ - cuts f in one point, where it crosses f. We direct this pseudoline in the direction from the negative to the positive side of f. This direction induces an **increasing direction** on the edge.

Pseudolinear Programming

A **solution** to a pseudolinear program $\mathcal{P} = (\mathcal{A}; p, f)$ is a vertex v of the feasible region R, not in p, such that no edge of R incident to v is outgoing.

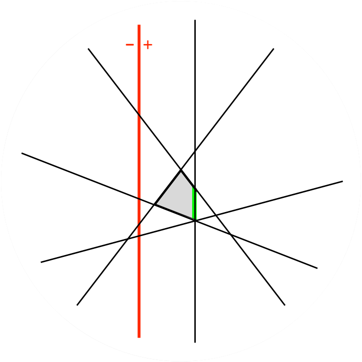
- **If the feasible region is non empty and bounded, then a pseudolinear program has at least one solution**.

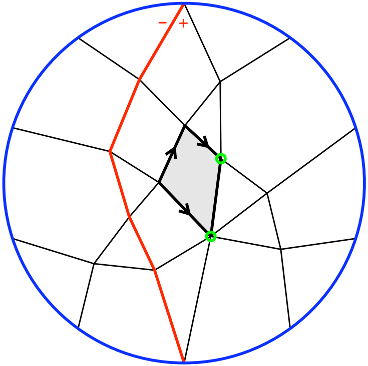
The statement asserting the existence of a solution of a pseudolinear program is the same as in the real case. However, its proof is less simple.

- **In pseudolinear programming, the graph of increasing directions may contain directed cycles**.

Basics of linear programming in oriented matroids are mainly due to R. Bland, J. Edmonds, K. Fukuda, J. Lawrence, A. Mandel (1977-80')

Pseudolinear Programming





A linear program in dimension 2, and its oriented matroid version as a pseudolinear program.

Tableau of a Basis

Let B be a matroid basis of a dimension-d pseudohyperplane arrangement, with set of hyperplanes E. We have $|B| = d + 1$.

The (fundamental) **tableau** of B is the $E \times E$ matrix with coefficients in $\{+, -, 0\}$ defined as follows.

A column $e \in B$ of the fundamental tableau of a basis B is the sign-vector of the positions $+, -, 0$ of the vertex $e^+ \cap \bigcap_{b \in B \setminus e} b$ with respect to the hyperplanes in E.

A column $e \notin B$ is a $-$ at (e, e) and 0 elsewhere.[2]

[2] In terms of oriented matroid theory, a column $e \in B$ is the sign-vector of the fundamental cocircuit $C^*(B; e)$ such that the sign at (e, e) is $+$. By oriented matroid orthogonality, a row $e \notin B$ is the opposite of the sign-vector of the fundamental circuit $C(B; e)$ such that the sign at (e, e) is a $-$.

Optimal Bases

An **optimal basis** of a d-dimensional pseudolinear program $\mathcal{P} = (\mathcal{A}; p, f)$ is a matroid basis B of $\mathcal{A} \cup \{f\}$ such that in the tableau of B

- $p \in B$, $f \notin B$,
- **the non zero signs of the column of p in the rows in $\mathcal{A}$ are plus.**[3]
- **the non zero signs of the row of f in the columns $\neq p$ are minus.**[4]

- **(The Simplex Criterion) A vertex v of the feasible region R is a solution to $\mathcal{P}$ if and only if there is an optimal basis $B = \{b_1 = p, b_2, \ldots, b_r\}_<$ such that $v = b_2 \cap b_3 \cap \ldots \cap b_r$.**

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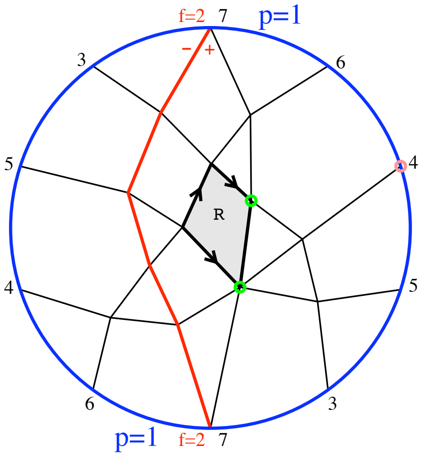
[3] this condition says that $v = \bigcap_{b \in B \setminus p} b$ is a vertex of the feasible region R.

[4] in the real case, this condition is equivalent to saying that the cone $\bigcap_{b \in B \setminus p} b^+$ (which contains R) is on the negative side of the hyperplane parallel to f through v.

Optimal Bases



$\mathcal{A}_7$

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	1	2	3	4	5	6	7
1	+						
2	+	−					−
3			+				
4	+		−	−			+
5	+		−		−		+
6	+		+			−	−
7							+

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	1	2	3	4	5	6	7
1	+						
2	+	−		−	+		
3	+		−	+	−		
4				+			
5					+		
6	+			−	+	−	
7				+	−		−

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	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−	−			+
4				+			
5				+	−		−
6	+			−		−	−
7							+

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	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−		−		+
4				−	+		+
5					+		
6	+				−	−	−
7							+

- **Optimal bases are not unique in general.**

Main Definition: Fully Optimal Bases

Let $\mathcal{A} = \{e_1, e_2, \ldots, e_n\}_<$ $(e_2 \neq e_1)$ be a **linearly ordered** arrangement of pseudohyperplanes in dimension d.

We define a **fully optimal basis** of $\mathcal{A}$ as an ordered basis $B = \{b_1, b_2, \ldots, b_r\}_<$ $r = d + 1$ of the oriented matroid $M(\mathcal{A})$ such that

- **the first sign $\neq 0$ in every row of the tableau of B is a plus,**
- **the first sign $\neq 0$ in every column $\neq e_1$ is a minus.**

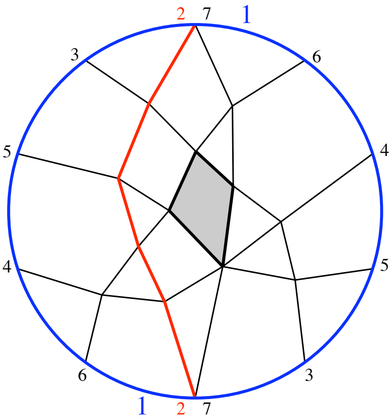
These two conditions immediately imply that

- $e_1 \in B$ **and** $e_2 \notin B$

It follows from this remark that

$\star$ **A fully optimal basis of an ordered hyperplane arrangement $\mathcal{A} = \{e_1, e_2, \ldots,\}$ is an optimal basis of the linear program $(\mathcal{A}; e_1, e_2)$ on the fundamental region of $\mathcal{A}$.**

A Fully Optimal Basis



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	1	2	3	4	5	6	7
1	+						
2	+	−					−
3			+				
4	+		−	−			+
5	+		−		−		+
6	+		+			−	−
7							+

147

	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−	−			+
4				+			
5				+	−		−
6	+			−		−	−
7							+

157

	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−		−		+
4				−	+		+
5					+		
6	+				−	−	−
7							+

⋆⋆ An ordered peudohyperplane arrangement with a bounded non-empty fundamental region has exactly one fully optimal basis.

LP Construction of the Fully Optimal Basis

(1) We first determine the **active vertex $\mathbf{v_1 = b_2 \cap b_3 \cap \ldots \cap b_r}$** of the fundamental region R, where $\{b_1 = e_1, b_2, \ldots, b_r\}_<$ is the fully optimal basis to be constructed.

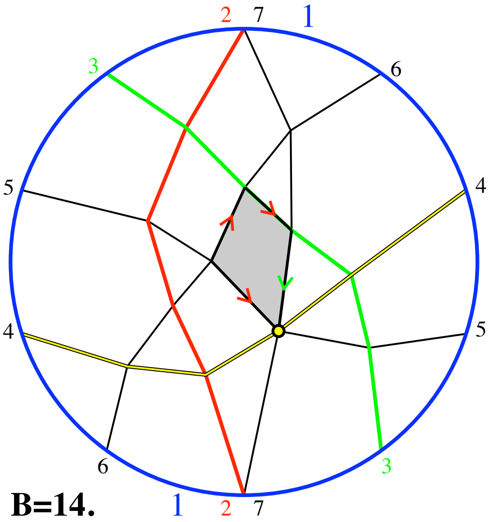
The vertex v_1 is a solution to the linear program defined on R_1, with plane at infinity $b_1 = e_2$ and objective function $f = e_2$ (supposing $e_1 \neq e_1$). However, this property does not suffice to determine v_1 in general.

$\star\star$ v_1 **is the unique solution of a lexicographic multiobjective program** (lexicographic multiprogram, for short) $\mathcal{P}_1$ **on** R **defined by the minimal basis** $B_{\min} = \{p = e_1, f_1 = e_2, \ldots, f_{r-1}\}_<$ **of** $\mathcal{A}$**.**

The multiprogram $\mathcal{P}_1$ is the problem of determining the set of vertices of R_1 maximizing f_1 over R, then the set of vertices R_2 maximizing f_2 among R_2, etc., until a unique vertex v_1 is obtained.

$\star\star$ b_2 **is the smallest hyperplane of** $\mathcal{A}$ **containing** v_1.

LP Construction of the Fully Optimal Basis



B=14.

LP Construction of the Fully Optimal Basis

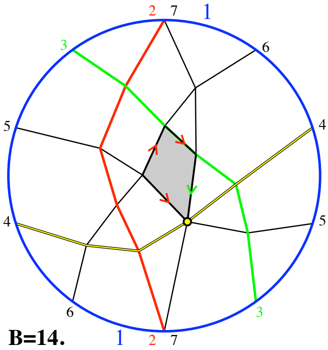
(2) The 2nd step is similar to the 1st, but concerns the edges of R containing v_1. By linearity, we may consider the projection from v_1 on the plane at infinity.

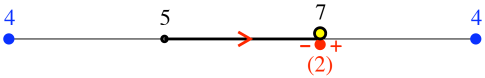
We define a 2nd lexicographic multiprogram $\mathcal{P}_2$ in the space $s_2 = p_1 \cap \overline{b_2^+} \approx \mathbf{B}^{d-1}$ with plane at infinity $p_2 = p_1 \cap b_2 \approx \mathbf{S}^{d-2}$. The hyperplanes of $\mathcal{A}_2$ are the traces on s_2 of the hyperplanes of $\mathcal{A}$ containing v_1. The objective functions of $\mathcal{P}_2$ are the traces on s_2 of the objective functions of $\mathcal{P}_1$. The feasible region R_2 is the projection of R_1 (and also the fundamental region of $\mathcal{A}_2$).

$\star\star$ $\mathcal{P}_2$ **has as a vertex** v_2 **as unique solution**
$\star\star$ b_3 **is the smallest hyperplane of** $\mathcal{A}$ **containing** v_1 **and** v_2,
$\star\star$ $< v_1, v_2 >= b_3 \cap b_4 \cap \ldots \cap b_r$.

(3,4,...) Iterate the construction of (2).

LP Construction of the Fully Optimal Basis





B=147

The objective functions not belonging to the hyperplane arrangement are indicated in the figures by putting their name between parentheses (like 2 in the right diagram).

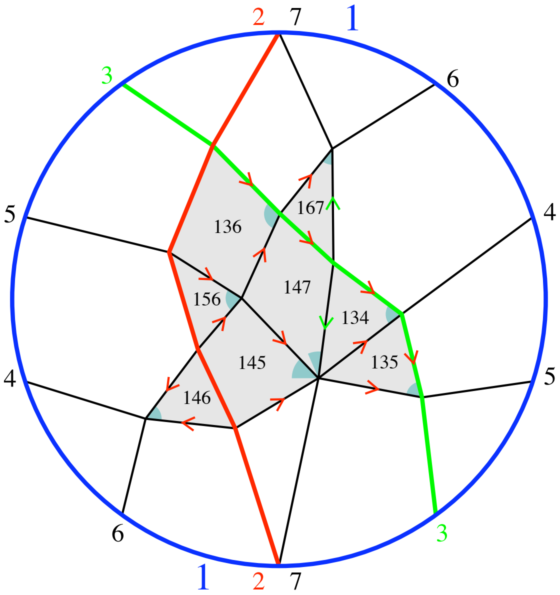
Main Theorem: The Active Bijection

Let $\mathcal{A}$ be an arrangement of (pseudo)hyperplanes, and B be a basis of $M(\mathcal{A})$. We recall that for $e \in B$ the fundamental cocircuit $C^*(B; e)$ is the set of hyperplanes of $\mathcal{A}$ not containing the vertex $\bigcap_{b \in B \setminus e} b$ (geometrically, the vertex opposite to e in the simplex B).

Suppose $\mathcal{A}$ is linearly ordered, $\mathcal{A} = \{e_1, \ldots\}_<$.
• A basis B of $M(\mathcal{A})$ is **internal** if for every $e \in \mathcal{A} \setminus B$ there exists $b \in B$ with $b < e$ and $e \in C^*(B; b)$.
• An internal basis B is **uniactive** if e_1 is the unique $e \in B$ which is the smallest element of its fundamental cocircuit $C^*(B; e)$.

⋆ **A fully optimal basis is internal and uniactive.**

⋆⋆⋆ **Fully optimal bases establish a bijection between the bounded non-empty regions of an ordered arrangement of pseudohyperplanes, and its uniactive internal bases.**



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The Tutte Polynomial

The active bijection defined for bounded regions in the previous pages is not sufficient to unify properties (1) and (2) of the introduction. We need to extend it to all regions.

For that purpose, we will use a classical tool of matroid theory, namely the Tutte polynomial. We recall that the Tutte polynomial of a matroid M on a set E can be defined by the formula

$$t(M; x, y) = \sum_{A \subseteq E} (x-1)^{r(M)-r_M(A)} (y-1)^{|A|-r_M(A)}$$

The rank of an hyperplane arrangement $\mathcal{A}$ is the codimension of its intersection. In dimension d, we have $r(\mathcal{A}) = d - \dim \bigcap_{e \in \mathcal{A}} e$.

The Tutte polynomial of the affine hyperplane arrangement $\mathcal{A}_7$ (running example) is

$t(\mathcal{A}_7; x, y) = y^4 + x^3 + 3y^3 + 4x^2 + 2xy + 6y^2 + 8x + 8y$

The Tutte Polynomial in Terms of B-Activities

Let B be a basis of a matroid M on a linearly ordered set E.

Theorem A [W.T. Tutte 1954 for graphs, extended to matroids by H. Crapo 1969]

$$t(M; x, y) = \sum_{B \text{ basis of } M} x^{\iota_M(B)} y^{\epsilon_M(B)}$$

$\iota_M(B) = \#$ **internally active** elements of M, i.e. $\#\ b \in B$ such that $b \leq e$ for all $e \in C^*(B; b)$.

$\epsilon_M(B) = \#$ **externally active** elements of M, i.e. $\#\ e \in E \setminus B$ such that $e < b$ for all $b \in B$ with $e \in C^*(B; b)$

The Tutte Polynomial in Terms of O-Activities

Let M be an oriented matroid on a linearly ordered set E.

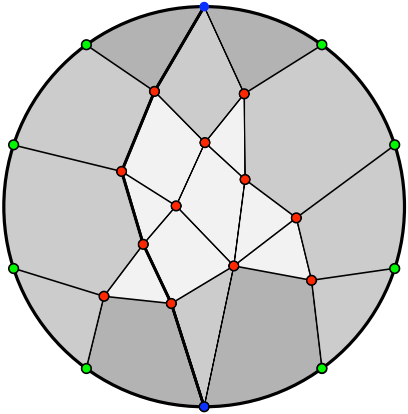
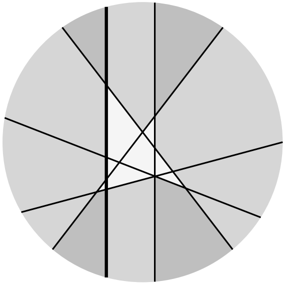
Theorem B [M. Las Vergnas 1982]

$$t(M; x, y) = \sum_{A \subseteq E} (\frac{x}{2})^{o^*(-_A M)} (\frac{y}{2})^{o(-_A M)}$$

where $-_A M$ denotes the reorientation on A of the oriented matroid M.

$o^*(M) = \#$ **dual-orientation active**, or O^*-active, elements of M, i.e. $\#$ smallest elements in some positive cocircuit of M.

$o(M) = \#$ **orientation active**, or O-active, elements of M, i.e. $\#$ smallest elements in some positive circuit of M.

⋆ **Geometrically, the dual-orientation activity of a region can be interpreted as the number of different infinity types of its vertices with respect to an ordering.**[5]

[5] In red, vertices at finite distance (not in 1). In green, vertices at simple infinity (in 1, but not in 2). In blue, vertices at double infinity (in both 1 and 2).

The Activity Relations

Comparing Theorems A and B we get the **orientation/basis activity relations**

- $o_{ij} = 2^{i+j} b_{ij}$ **for all** $i, j = 0, 1, \dots$

where
$o_{ij} = \#$ of reorientations with $o^* = i$ and $o = j$
$b_{ij} = \#$ of bases with $\iota = i$ and $\epsilon = j$.

- The active bijection for bounded regions is a bijective version of the relation $o_{10} = 2b_{10}$ [6], proved by T. Zaslavsky (1975) in the real case, and by M. Las Vergnas (1977) for oriented matroids.

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[6] $b_{10} = b_{01}$ is the β invariant of a matroid, a parameter first considered by H. Crapo in 1967.

In the case of $\mathcal{A}_7$ we have $\beta = 8$: there are 8 bounded regions.

The Active Mapping

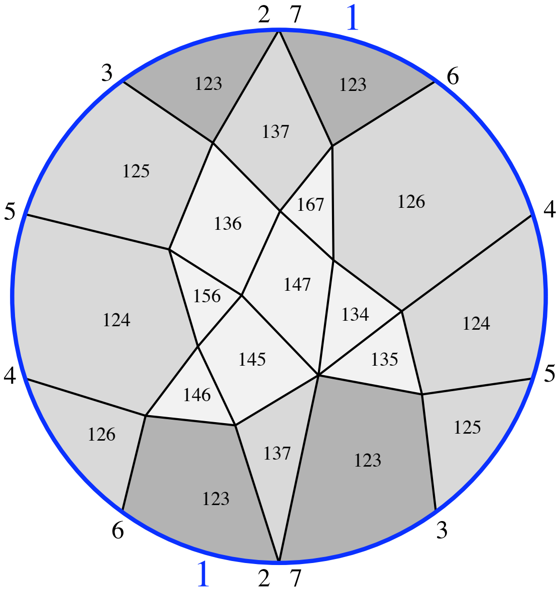
The **active orientation-to-basis mapping** is defined for all regions, and more generally for all reorientations of an oriented matroid. It can be considered as a 'bijective' version of the general activity relations.

$\star\star\star$ **The active mapping is activity preserving, with multiplicities given by activities.**

It would be too time-consuming to describe here into details the construction of the general active mapping. Roughly speaking, **we reduce to the bounded case by means of decomposing activities**. We first use duality to separate primal- and dual-orientation activities (by means of the oriented matroid Farkás Lemma). Then, we decompose into $(1, 0)$ and (0,1) activities. Finally, we glue together the various active bijections obtained.

Note that the factor 2 in $o_{10} = 2b_{10}$ accounts for the factor 2^{i+j} in the general activity relation.

The Active Mapping



The Active Mapping and Acyclic Orientations

The expression of the Tutte polynomial in terms of O-activities generalizes counting theorems of acyclic orientations in graphs (R. Stanley 1973), regions in hyperplane arrangements[7] (R.O. Winder 1966, T. Zaslavsky 1975), acyclic reorientations in oriented matroids (M. Las Vergnas 1975) [in order of increasing generality].

We have

$$\# \text{ of acyclic reorientations} = \sum_{i\geq 0} o_{i,0} = \sum_{i,j\geq 0} o_{ij} 1^i 0^j = t(2,0)$$

The active mapping provides a 'bijective' version of these counting theorems.

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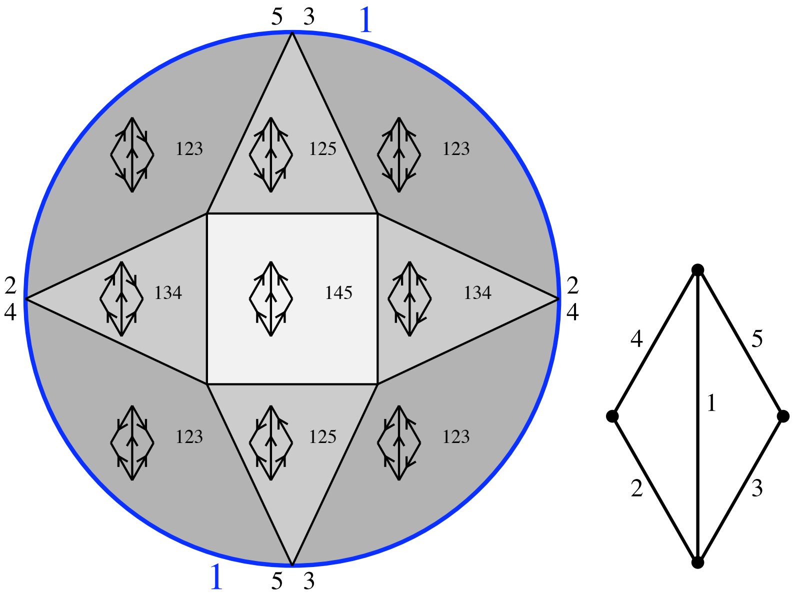
[7] The number of regions of a non central hyperplane arrangement is equal to half the number of acyclic reorientations of the oriented matroid of its affine dependencies.

For instance, $\mathcal{A}_7$ has $t(M(\mathcal{A}_7); 2, 0)/2 = (2^3 + 4.2^2 + 8.2)/2 = 20$ regions.



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5
1
2
3

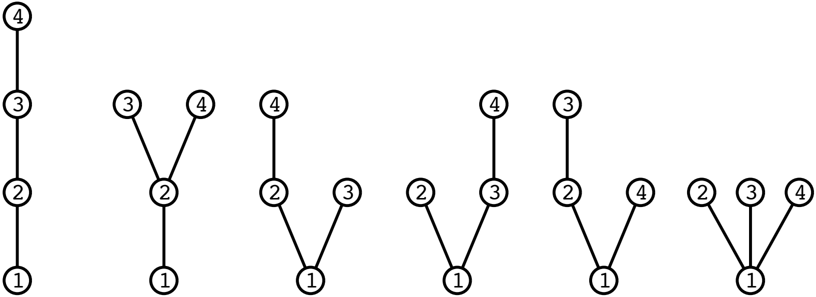
The Active Mapping
and the Permutation-to-Increasing-Tree Bijection

To relate the result (2) of the introduction to the active mapping, we consider the *braid arrangement* $\mathcal{B}_n$, defined by the $\binom{n}{2}$ hyperplanes h_{ij} in R^n with equations $-x_i + x_j = 0$ for $1 \leq i < j \leq n$.

It is folklore that **the regions of $\mathcal{B}_n$ corresponds bijectively to the permutations of the integers $1, 2, \ldots, n$.**

$\star\star$ **The active mapping restricted to the regions of $\mathcal{B}_n$ with respect to the colexicographic ordering, is equivalent to the classical bijection between $(n-1)$-permutations and increasing trees on n vertices.**

The Active Mapping for $\mathcal{B}_4$



123	132	213	231	312	321
1432	1342	1423	1243	1324	1234
2341	2431	2413	2143	2314	2134
		3142	3412	4132	3124
		3241	3421	4231	3214
					4123
					4213
					4312
					4321

Some Further Properties

(not presented in the talk)

• A direct algorithm to compute the region admitting a given uniactive internal basis as fully optimal basis

• Decomposition of activities and the active mapping

• Relationship between regions resp. reorientations mapped to a same basis by the active mapping, and properties of *active partitions*

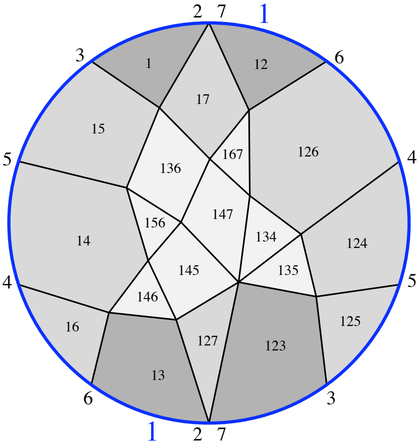
• Properties of the active mapping with respect to duality - oriented matroid duality and linear programming duality

• The *big active bijection*, a refinement of the active mapping

• Inductive properties

• Universality properties

The Big Active Bijection



The big active bijection - between reorientations and subsets - provides in particular a bijection between regions and NBC subsets, answering a question of the literature.

A Table of Bijections

structure	active bijection	
oriented matroid	active classes of reorientations	bases
	a.c. acyclic reorientations	internal bases
	a.c.. totally cyclic reorientations	external bases
	(opp. pairs) bounded acyclic reor.	uniactive internal bases
	reorientations	subsets
	acyclic reorientations	NBC subsets
uniform o.m	bounded regions	LP optimal vertices
hyperplane arr.	*reorientations* = *signatures*	*bases* = *simplices*
	acyclic reorientations = *regions*	
braid arrangement	permutations	increasing trees
hyperoctahedral arr.	pos. act. signed permutations.	increasing signed trees
(connected) graph	*reorientations* = *orientations*	*bases* = *spanning trees*
	unique sink acyclic orientations	internal spanning trees
	bipolar orientations	uniactive internal spanning trees

April 5, 2009