Simplifying with Extended Formulations

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Pretty Structure, Existential Polytime and Polyhedral Combinatorics
Paris, April 7–9, 2009

What is an extended formulation?

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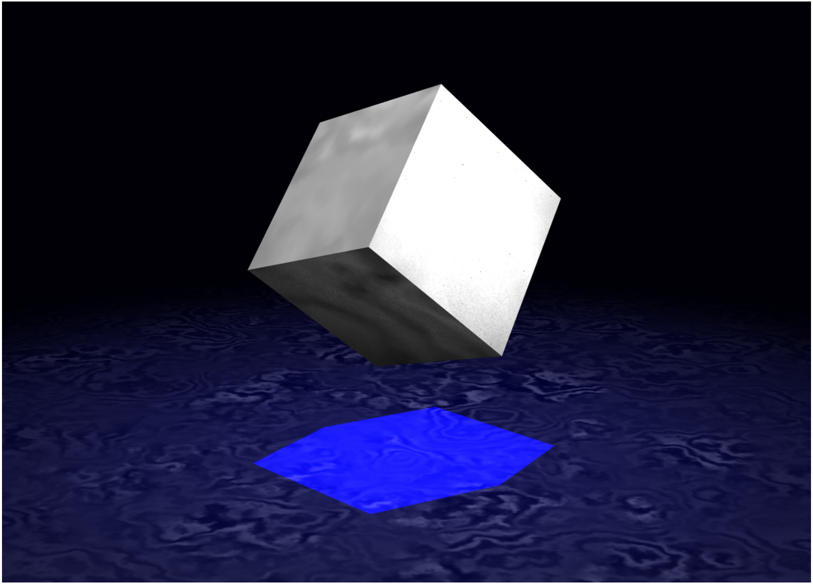

What is an orbitope?

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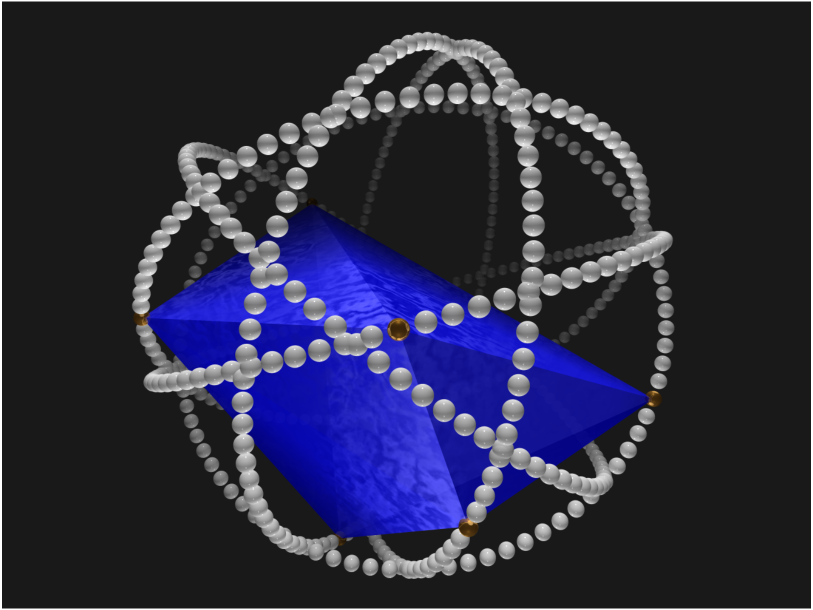
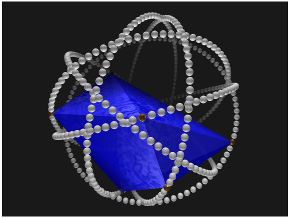


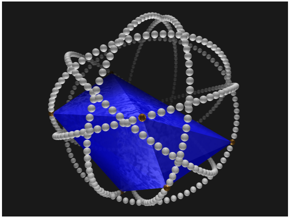
What makes up the orbits?

0	1	0	1	1	1	0	0	1
1	0	0	0	0	0	1	1	0
0	1	0	0	0	0	1	0	0
0	0	0	1	1	1	0	0	1
0	0	1	1	0	0	0	0	1
1	0	0	0	1	0	1	1	1
1	1	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	1



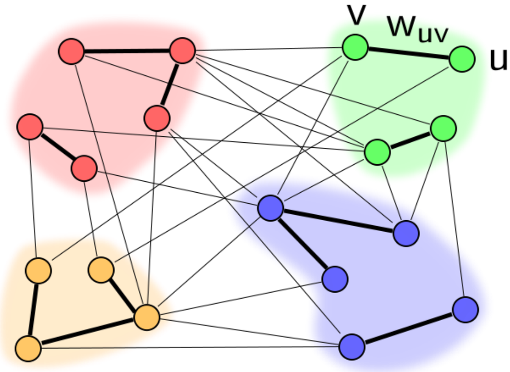
What are the representatives?

1	1	1	1	1	0	0	0	0
0	0	0	0	0	1	1	1	0
1	1	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0	0
0	0	1	1	0	0	0	0	1
1	0	1	0	1	0	1	1	0
0	1	0	1	0	0	1	0	0
0	0	1	0	0	0	0	0	0



Why?

- Symmetry breaking in certain integer programs



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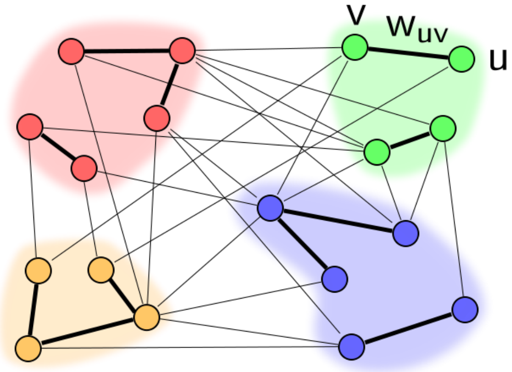

- Nice polytopes

Most Simple: Orbisacks

1	1
0	0
1	1
1	0
0	1
1	1
1	0
0	1

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- Special Knapsack:

  binary expansion first column

  $\geq$

  binary expansion second column

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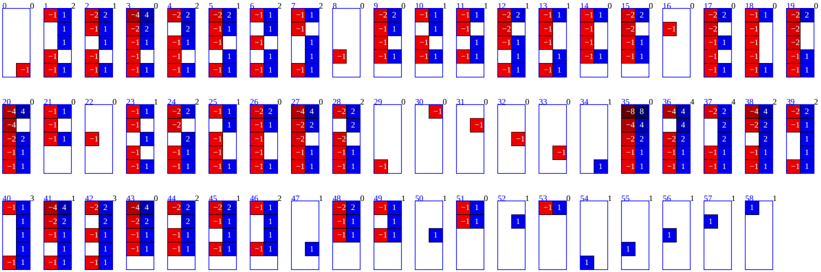
- Optimization in polynomial time

What polymake says

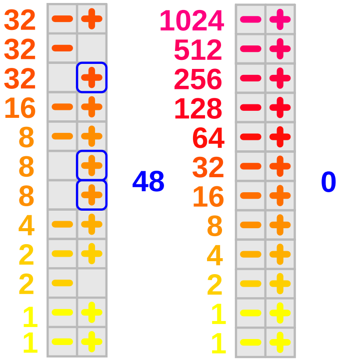


GBE inequalities

32	−	+
32	−	
32		+
16	−	+
8	−	+
8		+
8		+
4	−	+
2	−	+
2	−	
1	−	+
1	−	+

48

GBE inequalities



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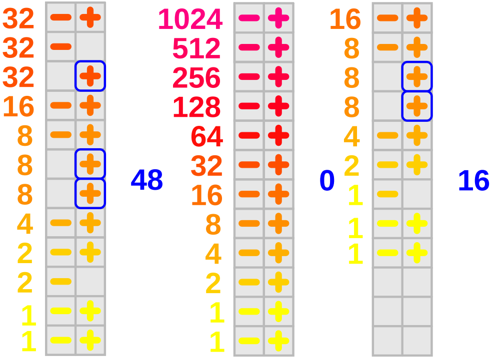
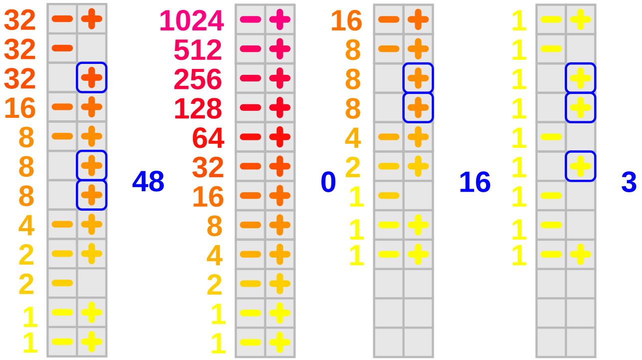


GBE inequalities



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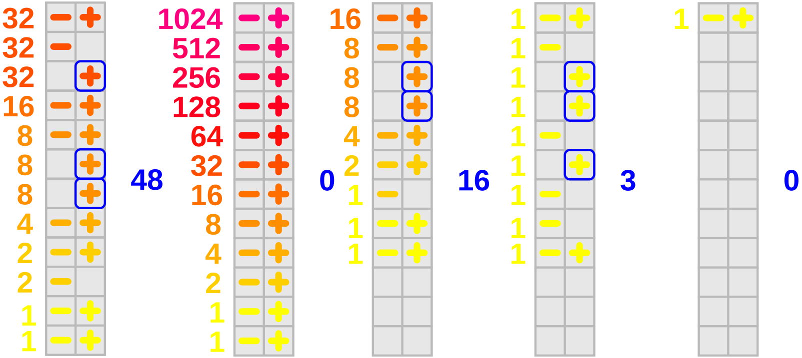
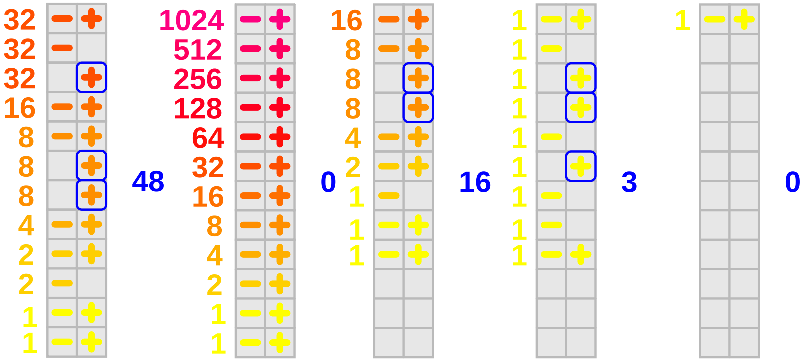


GBE inequalities



Theorem (K & Loos 07)

The exponentially many GBE inequalities and trivial inequalities form ideal descriptions of orbisacks.

Extending the vertices

1	1
0	0
1	1
1	0
0	1
1	1
1	0
0	1

0	0	0	1
0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	0
1	1	0	0
1	0	0	0
0	1	0	0

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1	1
0	0
1	1
1	0
0	1
1	1
1	0
0	1

0	0	0	1
0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	0
1	1	0	0
1	0	0	0
0	1	0	0

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0	0
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0	1
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1	0
0	1

0	0	0	1
0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	0
1	1	0	0
1	0	0	0
0	1	0	0

Extending the vertices

1	1
0	0
1	1
1	0
0	1
1	1
1	0
0	1

0	0	0	1
0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	0
1	1	0	0
1	0	0	0
0	1	0	0

Extending the vertices

1	1
0	0
1	1
1	0
0	1
1	1
1	0
0	1

0	0	0	1
0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	0
1	1	0	0
1	0	0	0
0	1	0	0

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1	1
0	0
1	1
1	0
0	1
1	1
1	0
0	1

0	0	0	1
0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	0
1	1	0	0
1	0	0	0
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0	0	0	1
0	0	1	0
0	1	0	0
1	1	0	0
1	0	0	0
0	1	0	0

Inequalities for the extension

0	0	0	1
0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	0
1	1	0	0
1	0	0	0
0	1	0	0

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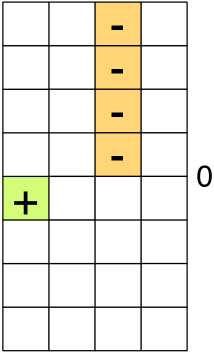
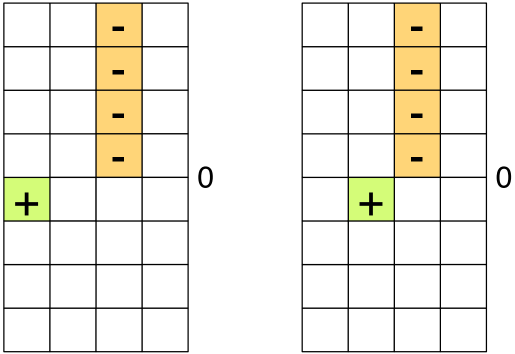
0	0	0	1
0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	0
1	1	0	0
1	0	0	0
0	1	0	0



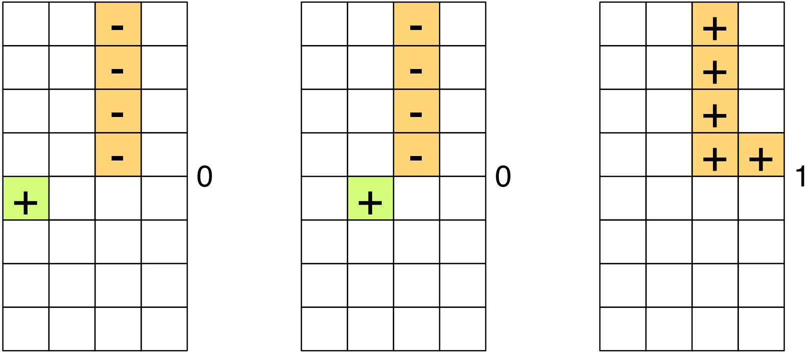
Inequalities for the extension

0	0	0	1
0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	0
1	1	0	0
1	0	0	0
0	1	0	0

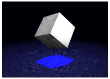


Inequalities for the extension

0	0	0	1
0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	0
1	1	0	0
1	0	0	0
0	1	0	0



Extended formulation for orbisacks

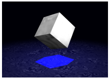


Theorem (K & Loos 08)

These inequalities (and bounds) yield extended formulations for Orbisacks with:

- $4p$ *variables*
- $3p$ *constraints*

Extended formulation for orbisacks



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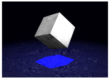
- Constraint matrix is totally unimodular.

Extended formulation for orbisacks



Theorem (K & Loos 08)

These inequalities (and bounds) yield extended formulations for Orbisacks with:

- $4p$ *variables*
- $3p$ *constraints*

- Constraint matrix is totally unimodular.
- Projections of integer points satisfying the formulation are the orbisack vertices.

General full orbitopes

Convex hulls of 0/1-matrices with lexicographically sorted columns (non-increasing).

1	1	1	1	1	0	0	0	0
0	0	0	0	0	1	1	1	0
1	1	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0	0
0	0	1	1	0	0	0	0	1
1	0	1	0	1	0	1	1	0
0	1	0	1	0	0	1	0	0
0	0	1	0	0	0	0	0	0

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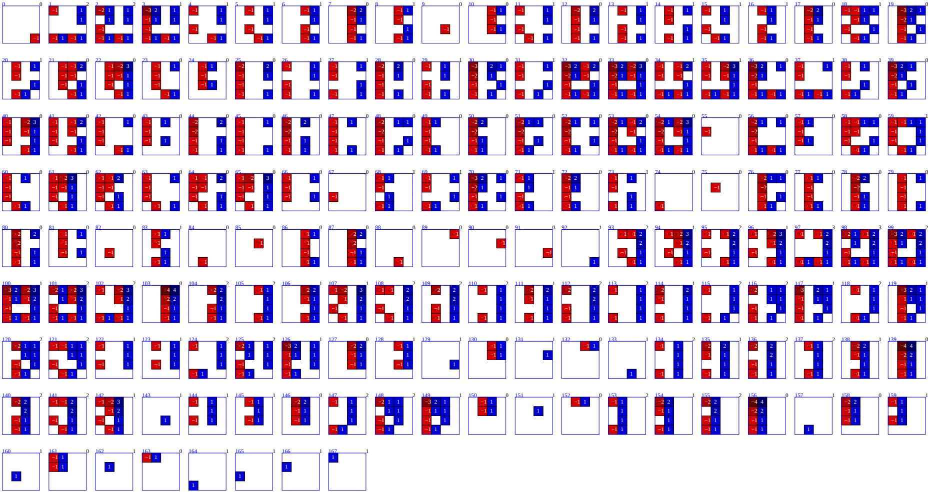
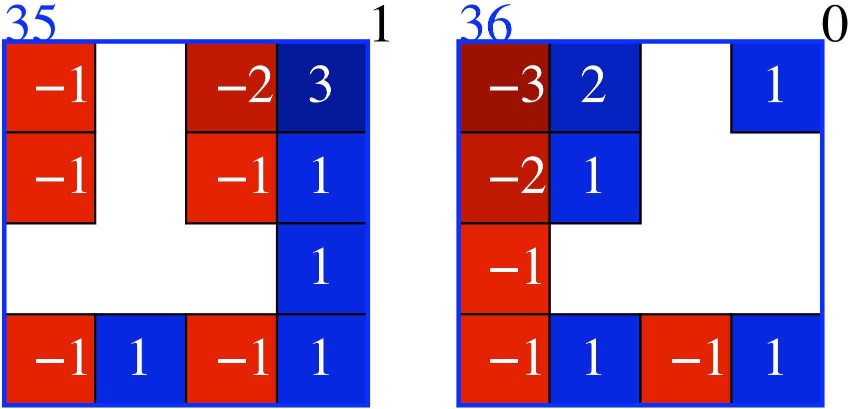
1	1	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0	0
0	0	1	1	0	0	0	0	1
1	0	1	0	1	0	1	1	0
0	1	0	1	0	0	1	0	0
0	0	1	0	0	0	0	0	0

- Optimization in polynomial time (K & Loos 08)

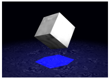
What polymake says



For instance



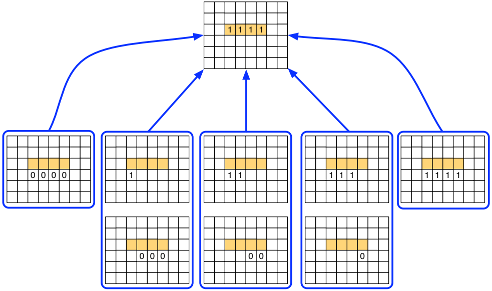
Extended formulation



Theorem (K & Loos 08)

Compact extended formulation for full orbitopes.

- Based on paths in a directed hypergraph.



Extended formulation

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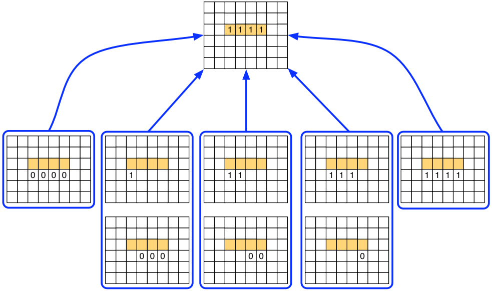


- Kipp Martin, Rardin, Campbell 90

Packing-/partitioning orbitopes

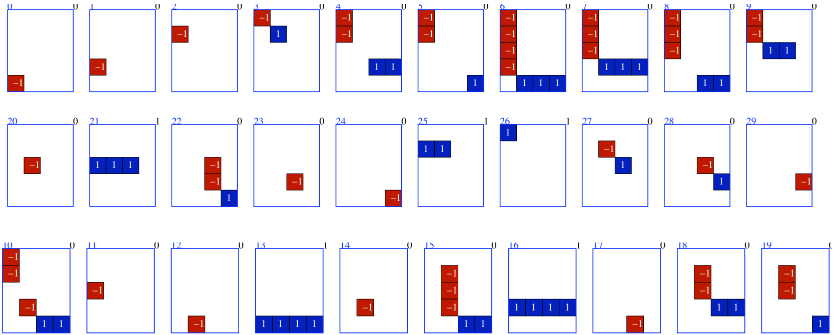
$O^{\leq}_{p,q}$ and $O^{=}_{p,q}$

Convex hull of all 0/1-matrices with at most/exactly one 1 per row and lexicographically sorted columns.

0					
1	0				
0	1	0			
0	0	0	0		
1	0	0	0	0	
0	0	1	0	0	0
0	0	0	1	0	0
0	1	0	0	0	0

1					
1	0				
0	1	0			
0	1	0	0		
1	0	0	0	0	
0	0	1	0	0	0
0	0	0	1	0	0
0	1	0	0	0	0

What polymake says

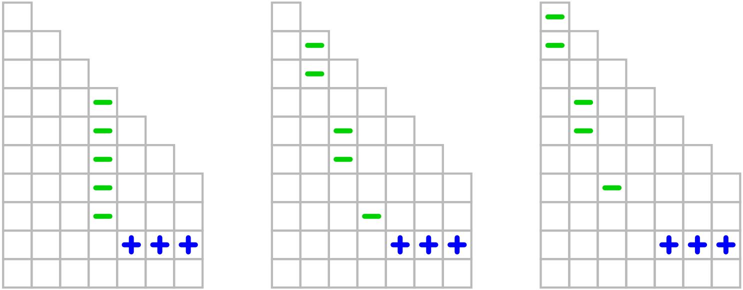


Some facts

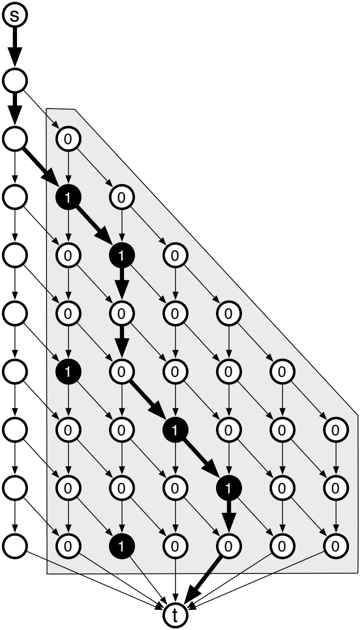
- $O^{=}_{p,q}$ is a face of $O^{\leq}_{p,q}$

Some facts

- $\mathrm{O}^{=}_{p,q}$ is a face of $\mathrm{O}^{\leq}_{p,q}$
- Complete description with exponentially many *shifted column inequalities* (K & Pfetsch 05)



Extending matrices by paths



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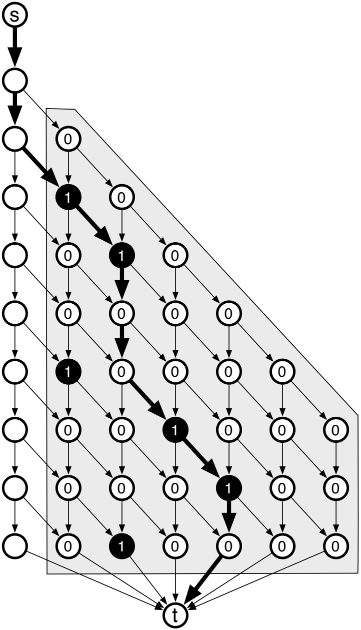

- x: node variables (matrix entries)

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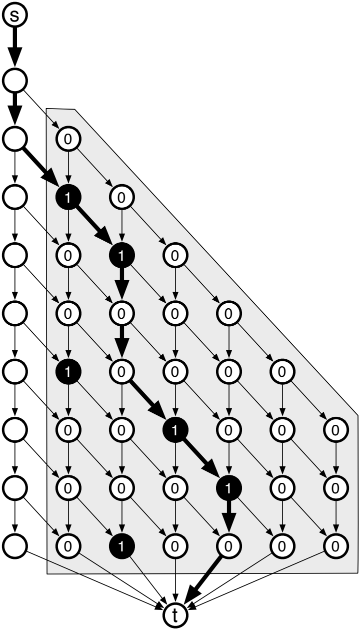

- x: node variables (matrix entries)
- y: arc variables

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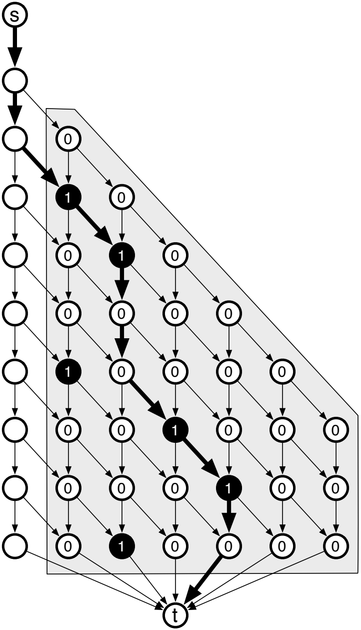

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- y: arc variables
- *feasible flow*: s-t-flow of value one

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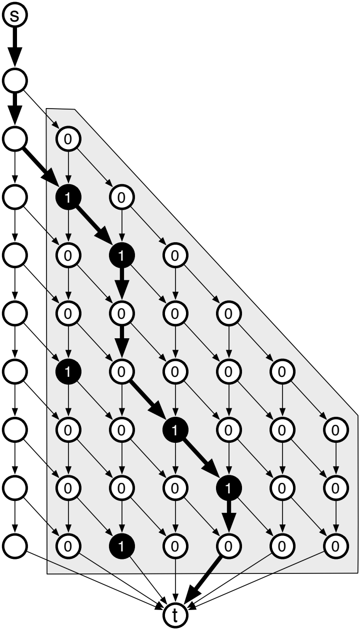

- x: node variables (matrix entries)
- y: arc variables
- *feasible flow*: s-t-flow of value one
- (Matrix cannot be recovered from s-t-path.)

The path formulation

Theorem (Faenza & K 08)

The convex hull of the path-extensions (x, y) *of the vertices* x *of* $\mathrm{O}^{\leq}_{p,q}$ *is the polytope* P *described by*

1. *the flow constraints on* y *and*

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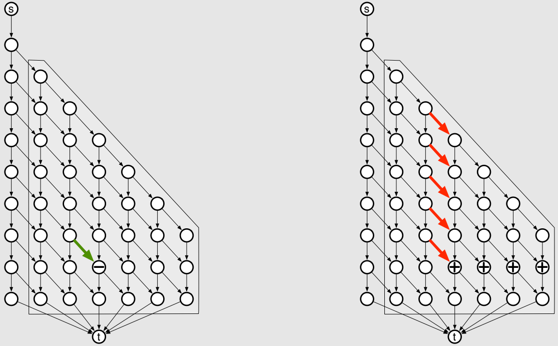
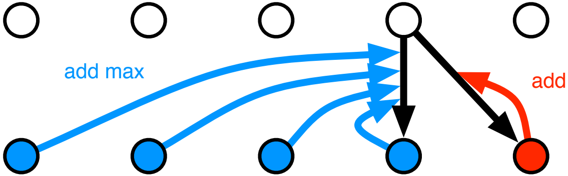
2. 

Proof: Show integrality of P

- For objective function $\langle c, (x, y) \rangle$ modify c to $\tilde{c}$:



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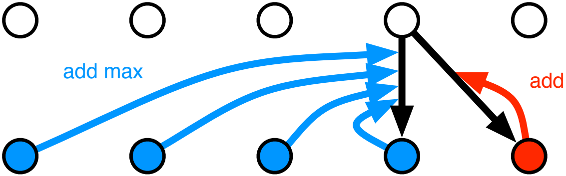

- For each $(x, y) \in P$: $\langle c, (x, y) \rangle \leq \langle \tilde{c}, (\mathbf{0}, y) \rangle$

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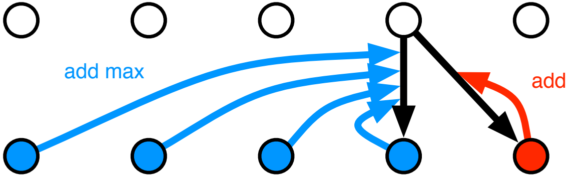

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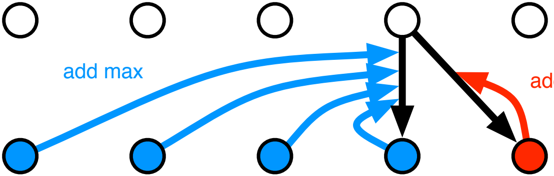

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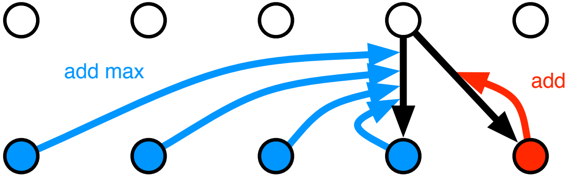
Proof: Show integrality of P

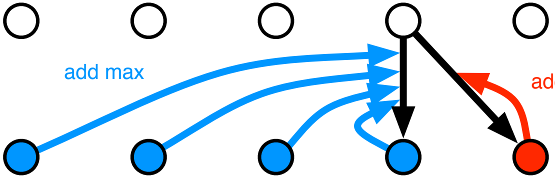
- For objective function $\langle c, (x, y)\rangle$ modify c to $\tilde{c}$:



- For each $(x, y) \in P$: $\langle c, (x, y)\rangle \leq \langle \tilde{c}, (\mathbf{0}, y)\rangle$
- For each s-t-path y there is a $0/1$ x with
  $(x, y) \in P$ and $\langle c, (x, y)\rangle = \langle \tilde{c}, (\mathbf{0}, y)\rangle$

Proof: Show integrality of P

- For objective function $\langle c, (x, y) \rangle$ modify c to $\tilde{c}$:



- For each $(x, y) \in P$: $\langle c, (x, y) \rangle \leq \langle \tilde{c}, (\mathbf{0}, y) \rangle$
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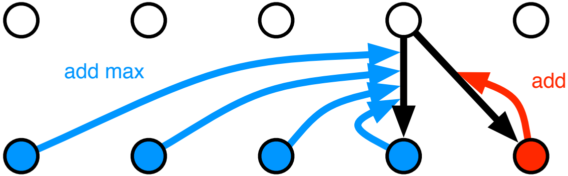
  $(x, y) \in P$ and $\langle c, (x, y) \rangle = \langle \tilde{c}, (\mathbf{0}, y) \rangle$
- Thus, P is integral.

Proof: Show integrality of P

- For objective function $\langle c, (x, y) \rangle$ modify c to $\tilde{c}$:

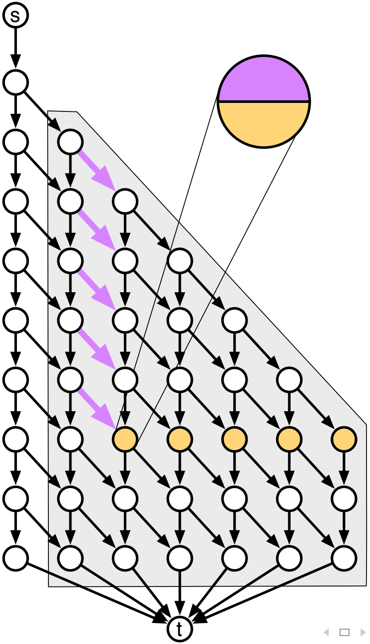


- For each $(x, y) \in P$: $\langle c, (x, y) \rangle \leq \langle \tilde{c}, (\mathbf{0}, y) \rangle$
- For each s-t-path y there is a $0/1$ x with

$$(x, y) \in P \text{ and } \langle c, (x, y) \rangle = \langle \tilde{c}, (\mathbf{0}, y) \rangle$$

- Thus, P is integral.
- (From this, the theorem follows easily.)

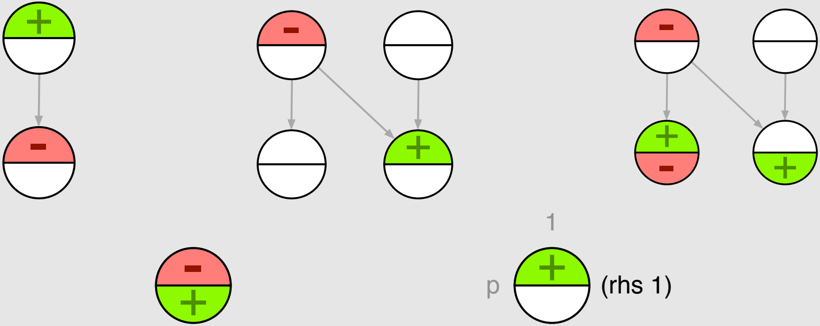
A linear transformation



A very compact formulation

Theorem (Faenza & K 08)

The following inequalities give an extended formulation for $O^{\leq}_{p,q}$:



($\approx$ $2pq$ variables, $4pq$ constraints, $10pq$ nonzeroes)

Deriving a description in the original space

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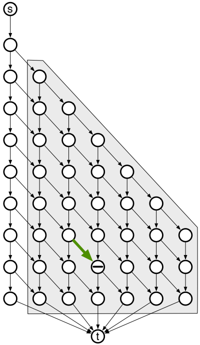

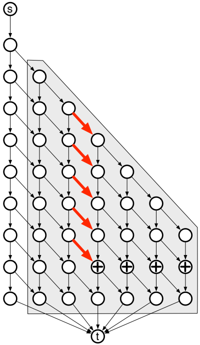

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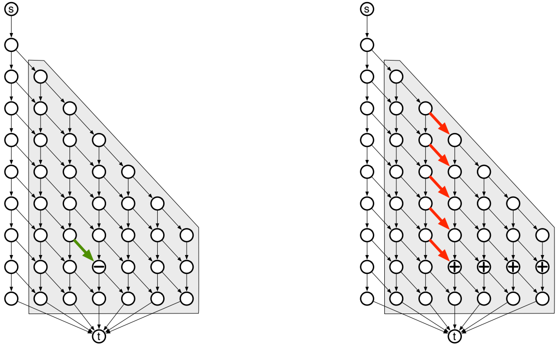
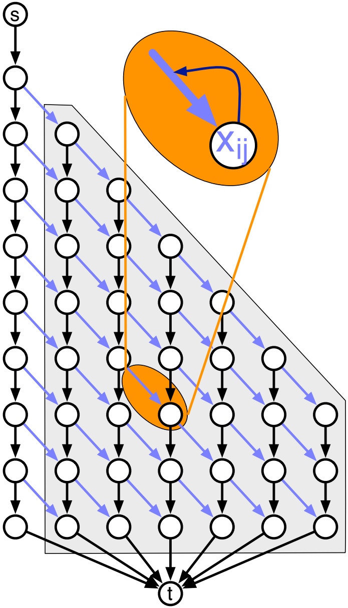
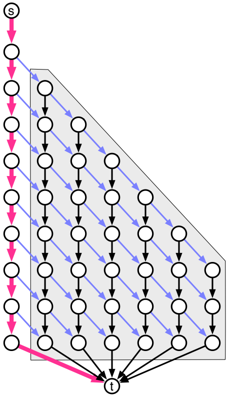


     hold)
   - and are valid for $O^{\leq}_{p,q}$.

Obtaining capacities from x



Constructing the right-most flow



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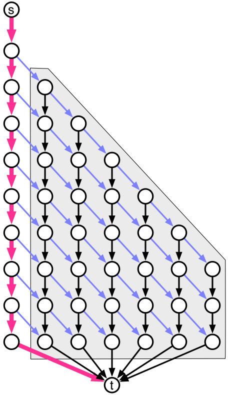

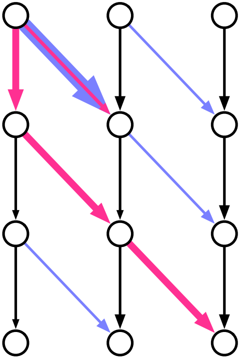

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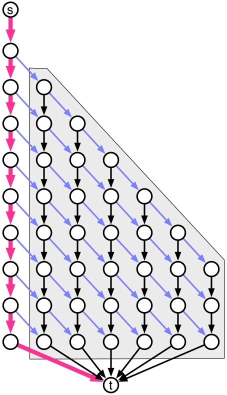

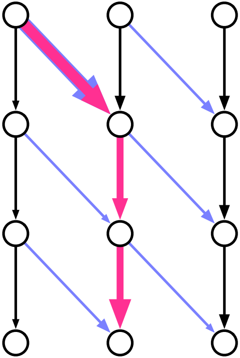

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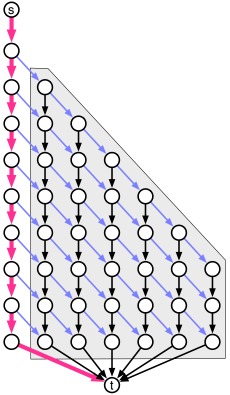

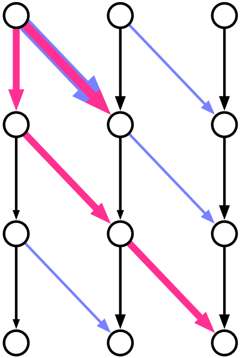

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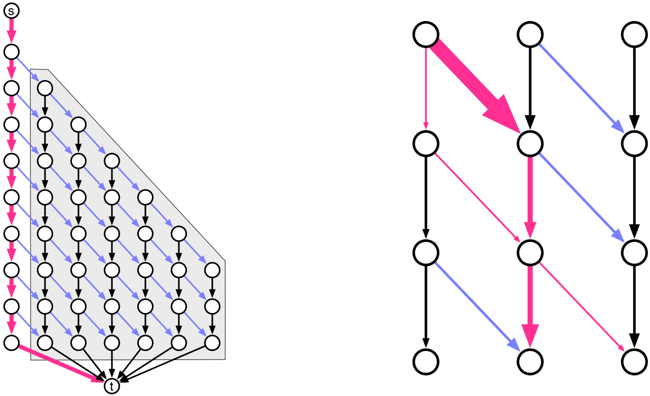

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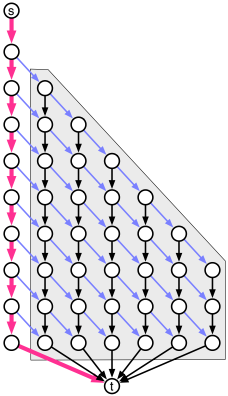
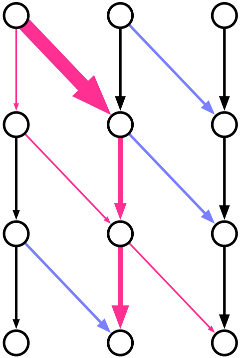
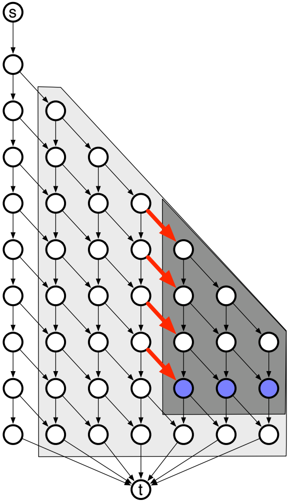




Canonical lifting of x

Vector (x, y) with *the* right-most feasible flow y.

SCIs come from left-most paths



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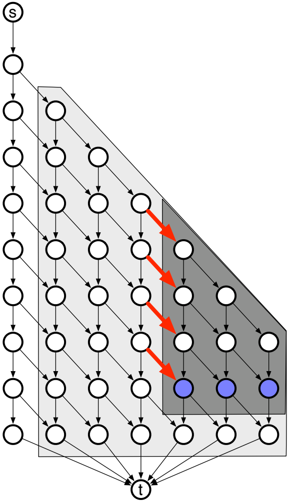

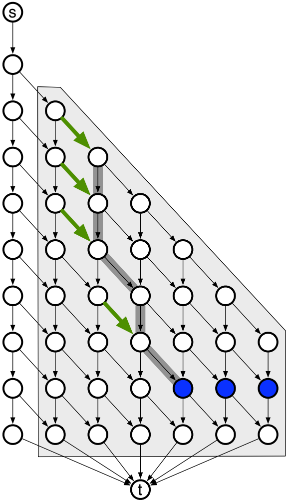

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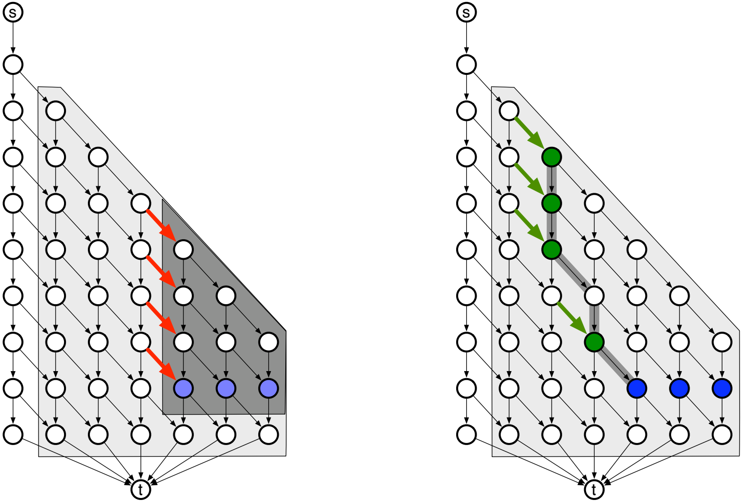

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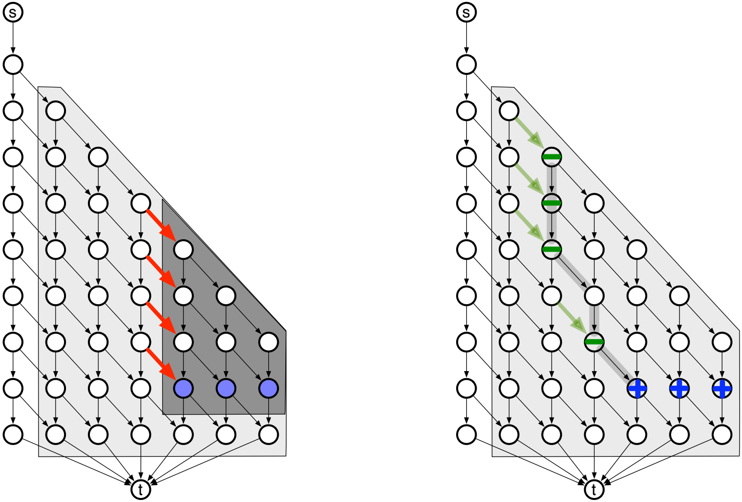
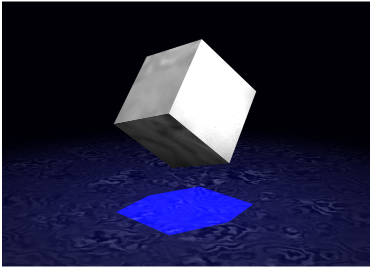
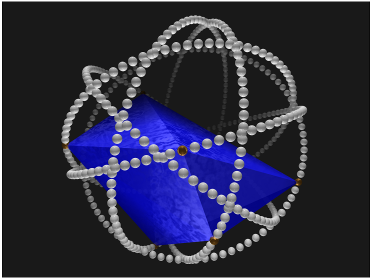


Extended formulations can . . .





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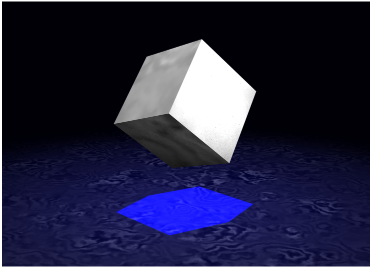

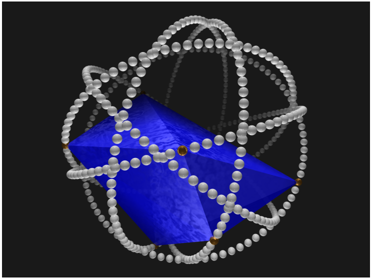

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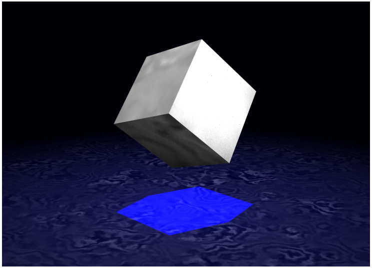

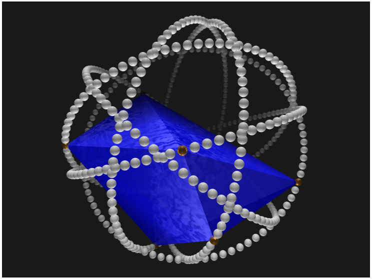

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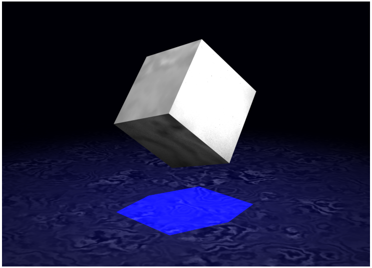

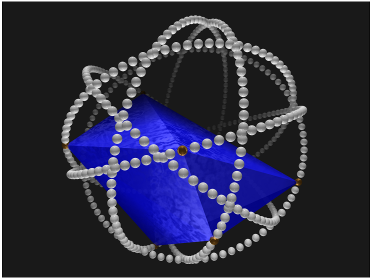

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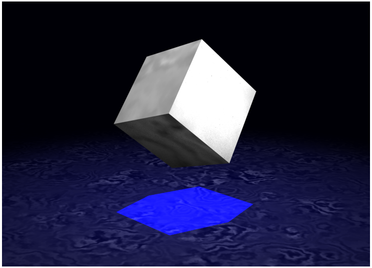

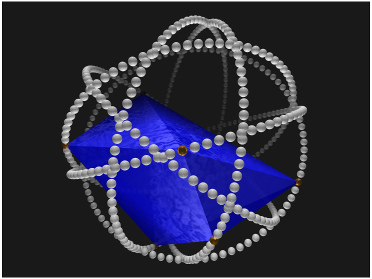

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Jac✓75