Crossing Minimization and Fun with Geometric Duality

Michael Jünger

Institut für Informatik, Universität zu Köln

Pretty Structure
Existential Polytime
Polyhedral Combinatorics

Latin Quarter, Paris

9 April 2009

Plan for the next 30 minutes

- 15 minutes on crossing minimization
- 05 minutes of fun with reminiscences
- 10 minutes of fun with geometric duality

Crossing Minimization

A very early J. R. Edmonds publication

A combinatorial representation for polyhedral surfaces, Notices American Mathematical Society 7 (1960), 643.

THE OCTOBER MEETING IN WORCESTER, MASSACHUSETTS
October 22, 1960

572-1. J. R. Edmonds: A combinatorial representation for polyhedral surfaces.

For any connected linear graph with an arbitrarily specified cyclic ordering of the edges to each vertex, there exists a topologically unique imbedding in an oriented closed surface so that the clockwise edge orderings around each vertex are as specified and so that the complement of the graph in the surface is a set of discs. (Received August 16, 1960.)

A very early J. R. Edmonds publication

A combinatorial representation for polyhedral surfaces, Notices American Mathematical Society 7 (1960), 643.

THE OCTOBER MEETING IN WORCESTER, MASSACHUSETTS
October 22, 1960

572-1. J. R. Edmonds: A combinatorial representation for polyhedral surfaces.

For any connected linear graph with an arbitrarily specified cyclic ordering of the edges to each vertex, there exists a topologically unique imbedding in an oriented closed surface so that the clockwise edge orderings around each vertex are as specified and so that the complement of the graph in the surface is a set of discs. (Received August 16, 1960.)

A very early J. R. Edmonds publication

A combinatorial representation for polyhedral surfaces, Notices American Mathematical Society 7 (1960), 646.

THE OCTOBER MEETING IN WORCESTER, MASSACHUSETTS
October 22, 1960

572-1. J. R. Edmonds: A combinatorial representation for polyhedral surfaces.

For any connected linear graph with an arbitrarily specified cyclic ordering of the edges to each vertex, there exists a topologically unique imbedding in an oriented closed surface so that the clockwise edge orderings around each vertex are as specified and so that the complement of the graph in the surface is a set of discs. (Received August 16, 1960.)

I'll tell you a little bit about

Available online at www.sciencedirect.com
ScienceDirect

ELSEVIER

Discrete Optimization 5 (2008) 373–388

DISCRETE OPTIMIZATION

www.elsevier.com/locate/disopt

A branch-and-cut approach to the crossing number problem☆

Christoph Buchheim[a], Markus Chimani[b], Dietmar Ebner[c], Carsten Gutwenger[b], Michael Jünger[a,*], Gunnar W. Klau[d,e], Petra Mutzel[b], René Weiskircher[f]

[a] Department of Computer Science, University of Cologne, Germany
[b] Department of Computer Science, University of Dortmund, Germany
[c] Institute of Computer Languages, Vienna University of Technology, Austria
[d] Department of Mathematics and Computer Science, Free University Berlin, Germany
[e] DFG Research Center MATHEON, Germany
[f] CSIRO Mathematical and Information Sciences, Melbourne, Australia

Received 31 January 2007; accepted 9 May 2007
Available online 5 November 2007

Paul Turán

Paul Turán 1910–1976

This goes back to 1944 . . .

JOURNAL OF GRAPH THEORY, VOL. 1, 7-9 (1977)

A Note of Welcome

PAUL TURÁN*
Budapest, Hungary

A note of welcome to the new *Journal of Graph Theory* might contain all sorts of good wishes and superficial praises of the beauty and usefulness of graph theory in general terms. My views on the latter, supported by facts, were given in [2]. As to the former, I can illustrate it better by giving some indications of the enchantment and help it gave me in the most difficult times of my life during the war.

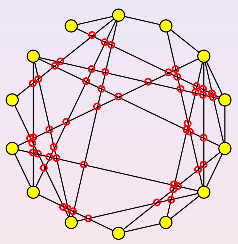
It sounds a bit incredible but it is true. The story goes back to 1940

This goes back to 1944 . . .

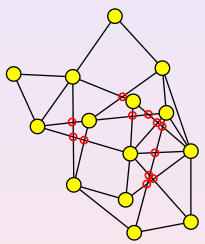
My next encounter with graph theory in these years was of a quite different nature. In July 1944 the danger of deportation was real in Budapest, and a reality outside Budapest. We worked near Budapest, in a brick factory. There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all the storage yards. The bricks were carried on small wheeled trucks to the storage yeards. All we had to do was to put the bricks on the trucks at the kilns, push the trucks to the storage yards, and unload them there. We had a reasonable piece rate for the trucks, and the work itself was not difficult; the trouble was only at the crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short this caused a lot of trouble and loss of time which was rather precious to all of us (for reasons not to be discussed here). We were all sweating and cursing at such occasions, I too; but *nolens-volens* the idea occurred to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized. But what is the minimum number of crossings? I realized after several days that the actual situation could have been improved, but the exact solution of the general problem with m kilns and n storage yards seemed to be very difficult and again I postponed my study of it to times when my fears for my family would end. (But the problem occurred to me again not earlier than 1952, at my first visit to Poland where I met Zarankiewicz. I mentioned to him my "brick-factory"-problem; he mentioned to me

Problem Definition and Motivation

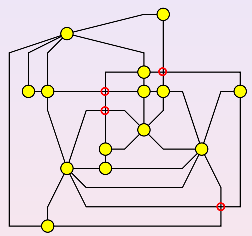
Given a graph $G = (V, E)$, draw it in two dimensions such that the number of crossings between its edges is minimum.



51 crossings



12 crossings



4 crossings

The crossing number cr(G) is the minimum number of such crossings for all two-dimensional drawings.

Problem Definition and Motivation

The crossing number problem

- was introduced by Turán in 1944 (for $K_{n,m}$)
- was shown to be NP-hard by Garey & Johnson [1983]
- is addressed heuristically in practice (planarization) or restricted to special drawings (bilayer, linear, circular)
- is unsolved even for very regular graph classes . . .

Crossing Number of Complete Graphs

- the crossing number of K_n is unknown in general
- the drawing rule of Zarankiewicz [1953] yields

$$Z(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

  crossings, hence $\mathrm{cr}(K_n) \leq Z(n)$
- it is conjectured that $\mathrm{cr}(K_n) = Z(n)$
- verified up to $n = 12$ by Pan & Richter [2007]
- recently, de Klerk et al. showed

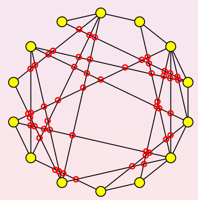
$$\lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{Z(n)} \geq 0.83$$

- similar situation for $K_{n,m}$

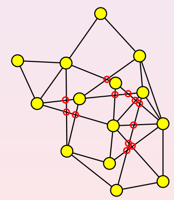
Applications

Applications for crossing minimization:

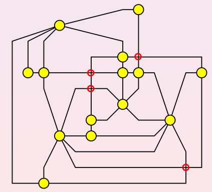
- design of a brick transport system on rails
  [crossings increase risk of accidents]
- VLSI design
  [crossings are expensive to realize]
- automatic graph drawing
  [crossings make the drawing less readable]



51 crossings



12 crossings



4 crossings

ILP Approach (First Attempt)

Our aim is to model the crossing number problem as an ILP.

Straightforward approach:

- introduce binary variable x_{ef} for each $\{e, f\}$ with $e, f \in E$
- interpret $x_{ef} = 1$ as "edge e crosses edge f"
- minimize $\sum x_{ef}$

Problem: checking feasibility is NP-complete!

Realizability

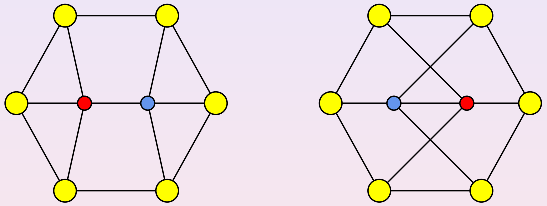
Problem:
Given $D \subseteq E \times E$, decide whether D is realizable, i.e., whether a drawing of G exists with e crossing f iff $(e, f) \in D$.

NP-complete by Kratochvíl [1991]

No hope for a useful ILP model with this choice of variables!

Realizability

Realizability depends on the **order** of crossings on an edge:



Number of potential orders is exponential...
It's **not** enough to determine the crossing edge pairs.

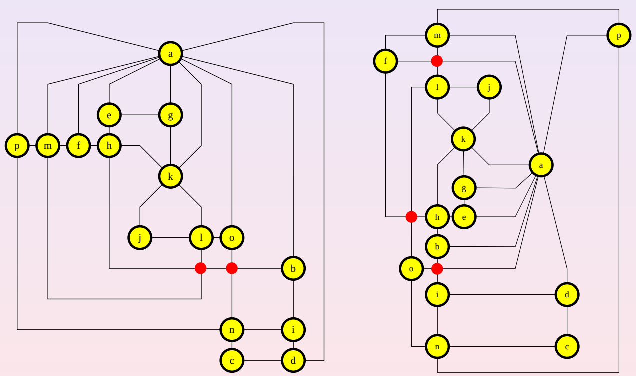
Crossing Restricted Drawings

To avoid this problem, consider crossing restricted drawings (CR-drawings):
allow at most one crossing per edge

However...

- optimum CR-drawings can have more than cr(G) crossings

Crossing Restricted Drawings



Crossing Restricted Drawings

To avoid this problem, consider crossing restricted drawings (CR-drawings):
allow at most one crossing per edge

However...

- optimum CR-drawings can have more than cr(G) crossings
- for dense graphs, CR-drawings don't even exist

Solution: replace every edge of G by a path of length $|E|$

Then a crossing-minimum CR-drawing of the resulting graph

- exists and has cr(G) crossings
- can be easily transformed into a drawing of G with the same number of edge crossings

ILP Approach (Second Attempt)

Search for a crossing-minimum CR-drawing of G:

- introduce binary variable x_{ef} for each $\{e, f\}$ with $e, f \in E$
- interpret $x_{ef} = 1$ as "edge e crosses edge f"
- minimize $\sum x_{ef}$
- introduce CR-constraints $\sum_{f \in E} x_{ef} \leq 1$

Realizability?!

Realizability

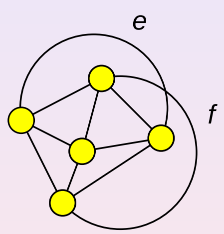
Call a set $D \subseteq E \times E$ crossing restricted if for all $e \in E$ there is at most one $f \in E$ with $(e, f) \in D$.

Problem:
Given a crossing restricted set $D \subseteq E \times E$,
decide whether D is realizable.

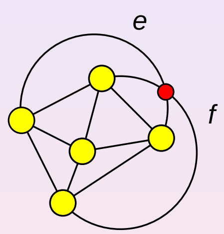
Can be done in linear time...

Realizability

Define G_D as the result of adding dummy nodes to G on every edge pair $(e, f) \in D$:



$G = (V, E)$, $D = \{(e, f)\}$



G_D

Construction is well-defined as D is crossing restricted!
I.e., D is realizable iff G_D is planar.

IP formulation ... Branch&Cut

This leads to

- an IP formulation in the x_{ef}-variable space,
- a separation heuristic that essentially amounts to Kuratowski-subgraph identification (linear time $O(|V| + |D|)$) by de Fraysseix & Ossona de Mendez [2003]).

Add some bells and whistles and get a branch&cut algorithm.

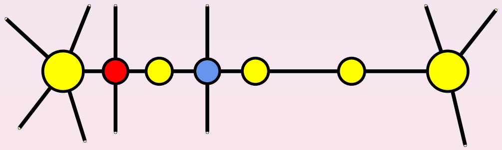
Experiments show that this approach

+ works
+ can solve benchmark instances up to $|V| = 40$
– can't solve dense instances
– produces a huge number of variables
– produces a lot of symmetry

Drawbacks of the Model

Replacing edges by paths. . .

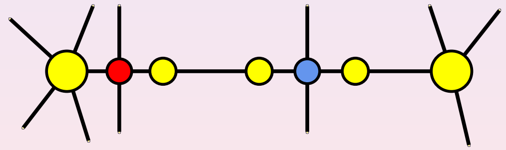
- yields up to $\Theta(|E|^4)$ variables in total
  [only cr(G) of them are 1 in an optimal solution]
- leads to many equivalent solutions:



Drawbacks of the Model

Replacing edges by paths. . .

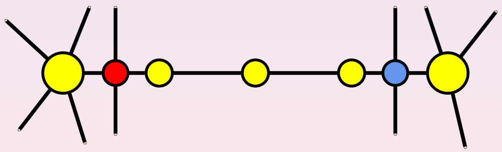
- yields up to $\Theta(|E|^4)$ variables in total
  [only cr(G) of them are 1 in an optimal solution]
- leads to many equivalent solutions:



Drawbacks of the Model

Replacing edges by paths. . .

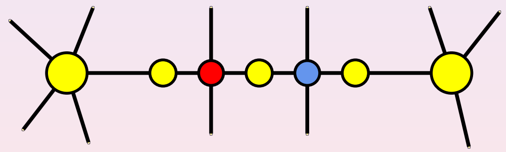
- yields up to $\Theta(|E|^4)$ variables in total
  [only cr(G) of them are 1 in an optimal solution]
- leads to many equivalent solutions:



Drawbacks of the Model

Replacing edges by paths. . .

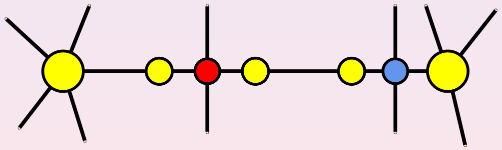
- yields up to $\Theta(|E|^4)$ variables in total
  [only cr(G) of them are 1 in an optimal solution]
- leads to many equivalent solutions:



Drawbacks of the Model

Replacing edges by paths. . .

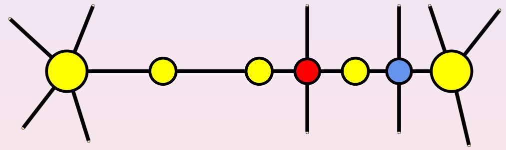
- yields up to $\Theta(|E|^4)$ variables in total
  [only cr(G) of them are 1 in an optimal solution]
- leads to many equivalent solutions:



Drawbacks of the Model

Replacing edges by paths. . .

- yields up to $\Theta(|E|^4)$ variables in total
  [only cr(G) of them are 1 in an optimal solution]
- leads to many equivalent solutions:



Solution to both problems is column generation!

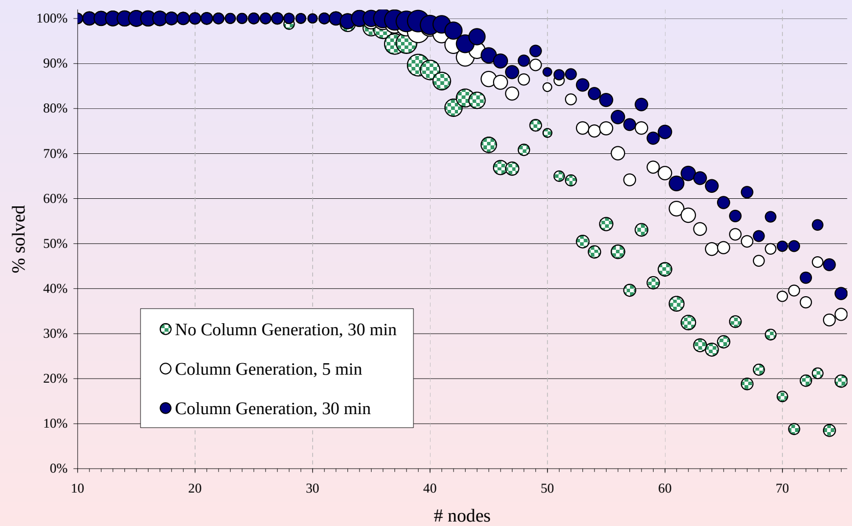
Computational Results

Test instances: Rome library of undirected graphs

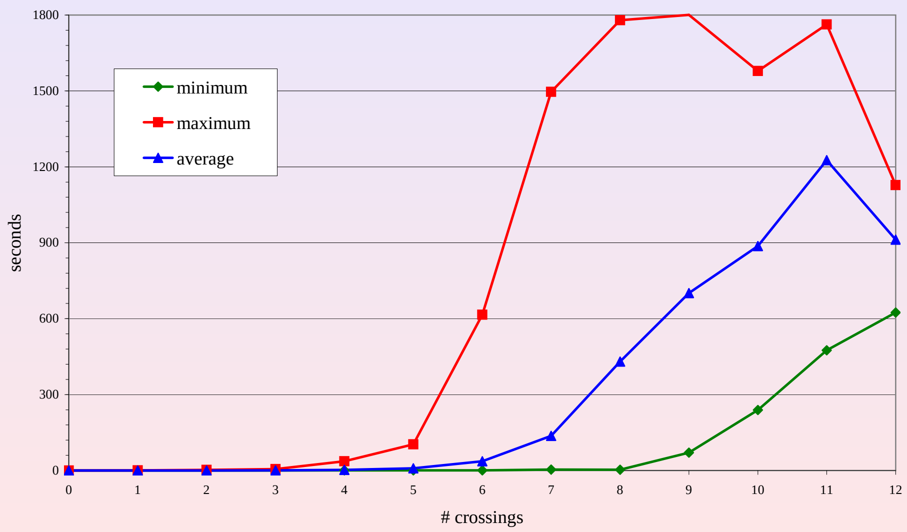
- standard benchmark set in automatic graph drawing
- derived from graphs arising in practical applications
- contains graphs on 10 to 100 nodes
- graphs are sparse

Time limit:
30 cpu minutes on an AMD Opteron with 2.4 GHz, 32 GB

Percentage of Solved Instances



Runtimes by Crossing Number



Complete Graphs

Using our general approach, we can solve K_8
(though its crossing number is 18)

Special approach for complete graphs:
use knowledge of $\mathrm{cr}(K_n)$ when computing $\mathrm{cr}(K_{n+1})$

Using this specialized approach, we can solve K_{12} . . .

Reminiscences

Bonn 1978

Gerd Reinelt and me students of CS and OR in Bonn. . .

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 71B, No. 4, October–December 1967

Systems of Distinct Representatives and Linear Algebra*

Jack Edmonds
Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(November 16, 1966)

Some purposes of this paper are: (1) To take seriously the term, "term rank." (2) To make an issue of not "rearranging rows and columns" by not "arranging" them in the first place. (3) To promote the numerical use of Cramer's rule. (4) To illustrate that the relevance of "number of steps" to "amount of work" depends on the amount of work in a step. (5) To call attention to the computational aspect of SDR's, an aspect where the subject differs from being an instance of familiar linear algebra. (6) To describe an SDR instance of a theory on extremal combinatorics that uses linear algebra in very different ways than does totally unimodular theory. (The preceding paper, Optimum Branchings, describes another instance of that theory.)

Key Words: Algorithms, combinatorics, indeterminates, linear algebra, matroids, systems of distinct representatives, term rank.

1. Introduction

The well-known concept of term rank [5, 6],[1] is shown here to be a special case of linear-algebra rank. This observation is used to provide a simple linear-algebra proof of the well-known SDR theorem. Except for familiar linear algebra, the paper is self-contained.

Incidentally to SDR's, an algorithm is presented for

However, here the word "transversal" will be used differently.)

3. Matrices of Zeros and Ones

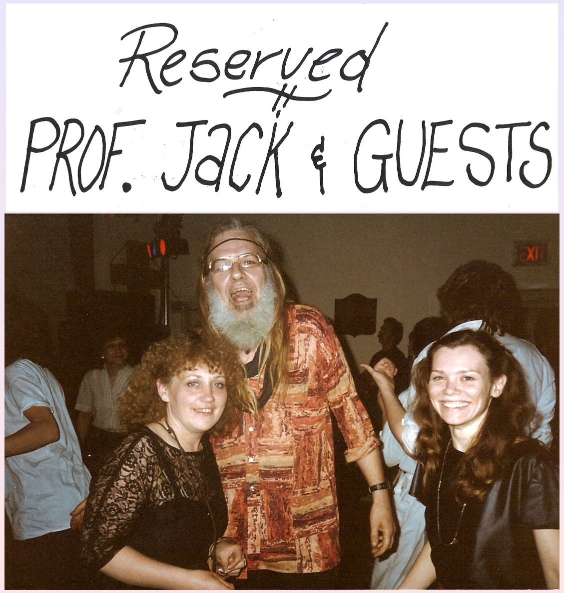
The subject of SDR's is frequently treated in the context of matrices of 0's and 1's. The *incidence matrix* of the family Q of subsets of E is the matrix

Bonn 1978

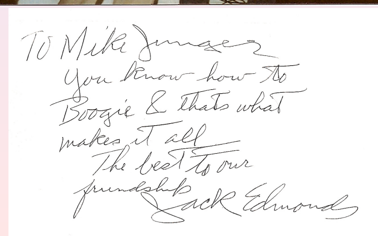
```
C***********************************************************************00000010
C***** EDMOND SUBPR. 15/06/78  VS. 2  JUENGER/REINELT *****************00000020
C***********************************************************************00000030
C*                                                                     *00000040
C* TESTCLASS :  2                                                      *00000050
C*                                                                     *00000060
C* COMPUTER :  IBM 370/168                                              *00000070
C*                                                                     *00000080
C* PROGRAM CATHEGORY :  9 (LINEAR ALGEBRAIC METHODS)                    *00000090
C*                                                                     *00000100
C* PROGRAM TITLE :  EDMOND                                              *00000110
C*                                                                     *00000120
C* AUTHORS :  G. REINELT / M. JUENGER                                   *00000130
C*                                                                     *00000140
C* CONTRIBUTOR :  A. BACHEM                                             *00000150
C*                                                                     *00000160
C* PURPOSE :  THIS FORTRAN-SUBROUTINE PERFORMS THE ALGORITHM OF JACK    *00000170
C*            EDMONDS WHICH COMPUTES RANK AND DETERMINANT OF ANY IN-   *00000180
C*            TEGER-MATRIX USING INTEGER-ARITHMETIC IN ORDER TO OBTAIN *00000190
C*            EXACT RESULTS                                            *00000200
.
.
.
C* VARIABLE NAMES :                                                    *00000490
C*   IROW, IROWNR : I*4 : ROW COUNTERS                                 *00000500
C*   ICOL, ICOLNR : I*4 : COLUMN COUNTERS                              *00000510
C*   ISTEP        : I*4 : STEP COUNTER                                 *00000520
C*   JACK         : I*4 : SPECIAL MATRIX ELEMENT TO BE CHOSEN BY THE   *00000530
C*                        ALGORITHM IN EACH STEP                       *00000540
C*   IDIVR        : I*4 : SPECIAL DIVISOR USED FOR THE COMPUTATIONS ON *00000550
C*                        THE MATRIX ENTRIES IN EACH STEP              *00000560
.
.
.
```

Bonn ISMP 1982

Southern Ontario Blues Association 1985



Southern Ontario Blues Association 1985

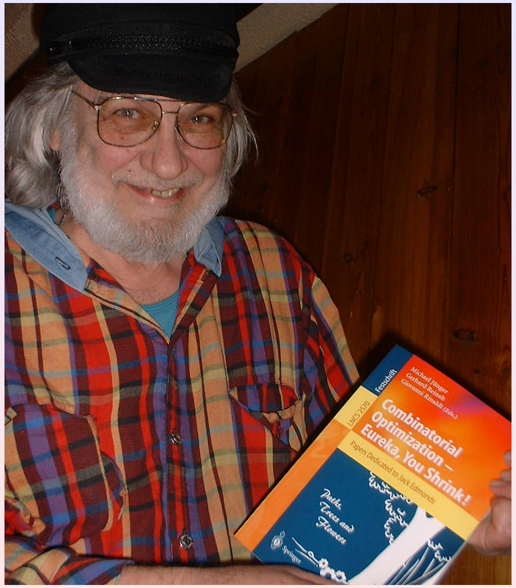


Augsburg 1989

Luminy, Summer 1990: Kathie, Jack & Alex

Aussois 2001: Jack preaching

Aussois 2002: Jack enjoying



Cologne 2004: Jack & Kathie with Pauline & Paul

Cologne 2004: Me and the Major

The Major

Movie

Fun with Geometric Duality

Two more early Jack Edmonds publications – the 1st

PATHS, TREES, AND FLOWERS

JACK EDMONDS

1. Introduction. A *graph* G for purposes here is a finite set of elements called *vertices* and a finite set of elements called *edges* such that each edge *meets* exactly two vertices, called the *end-points* of the edge. An edge is said to *join* its end-points.

A *matching* in G is a subset of its edges such that no two meet the same vertex. We describe an efficient algorithm for finding in a given graph a matching of maximum cardinality. This problem was posed and partly solved by C. Berge; see Sections 3.7 and 3.8.

Maximum matching is an aspect of a topic, treated in books on graph theory, which has developed during the last 75 years through the work of about a dozen authors. In particular, W. T. Tutte **(8)** characterized graphs which do not contain a *perfect* matching, or *1-factor* as he calls it—that is a set of edges with exactly one member meeting each vertex. His theorem prompted attempts at finding an efficient construction for perfect matchings.

Two more early Jack Edmonds publications – the 2nd

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 69B, Nos. 1 and 2, January–June 1965

Maximum Matching and a Polyhedron With 0,1-Vertices[1]

Jack Edmonds

(December 1, 1964)

A matching in a graph G is a subset of edges in G such that no two meet the same node in G. The convex polyhedron C is characterized, where the extreme points of C correspond to the matchings in G. Where each edge of G carries a real numerical weight, an efficient algorithm is described for finding a matching in G with maximum weight-sum.

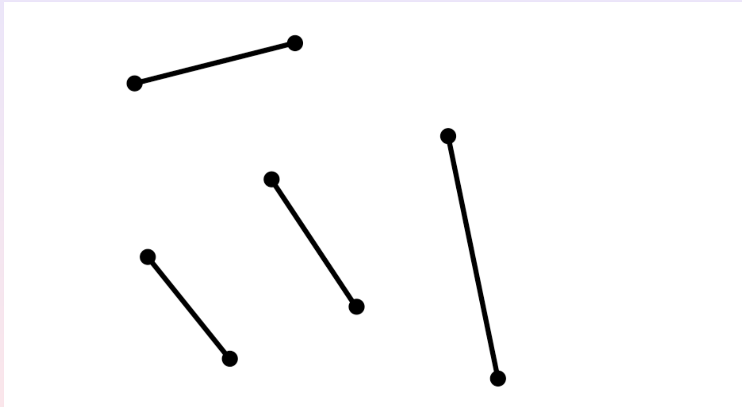
Section 1

An algorithm is described for optimally pairing a finite set of objects. That is, given a real numerical weight for each unordered pair of objects in a set Y, to select a family of mutually disjoint pairs the sum of whose weights is maximum. The well-known optimum assignment problem [5][2] is the special case where Y partitions into two sets A and B such that inequalities. In particular, we prove a theorem analogous to one of G. Birkhoff [1] and J. von Neuman [5] which says that the extreme points of the convex set of doubly stochastic matrices (order n by n) are the permutation matrices (order n by n). That theorem and the Hungarian method are based on Konig's theorem about matchings in bipartite graphs. Our work is related to results on graphs due to Tutte [4].

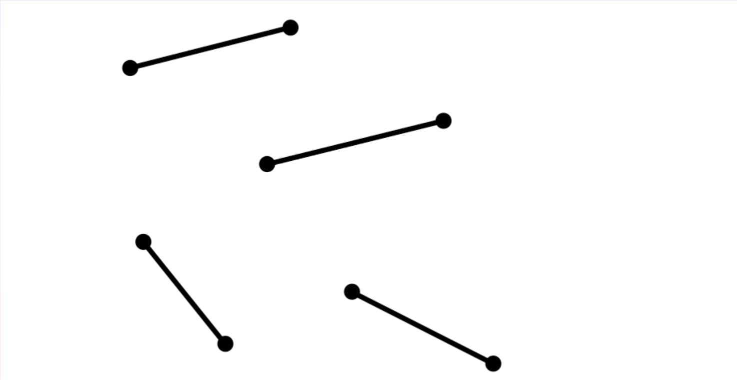
Written at the same time?

Movie

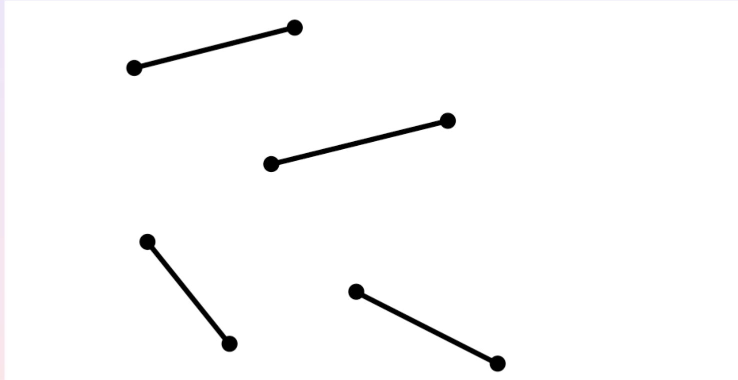
Prove that this perfect matching is not shortest . . .



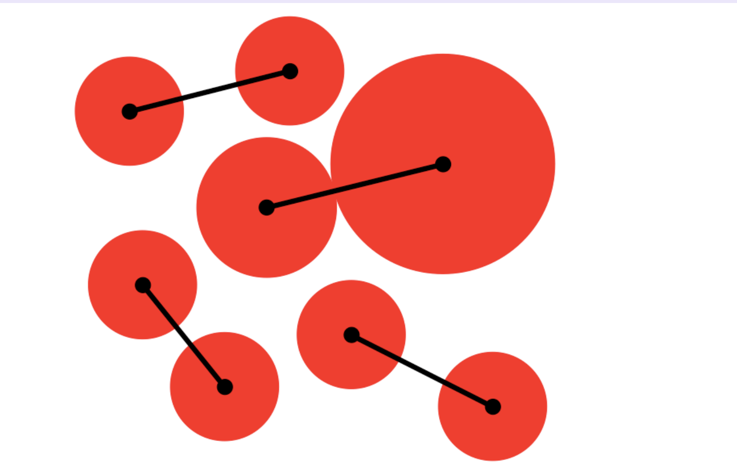
Show a shorter perfect matching!



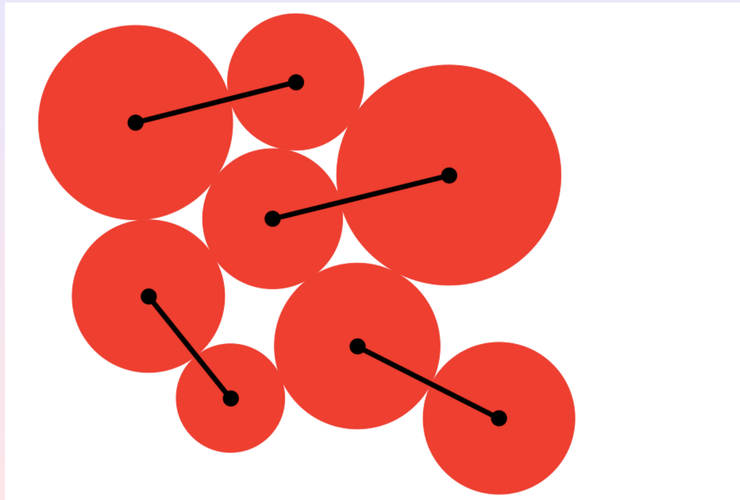
Show that this is a shortest perfect matching . . .



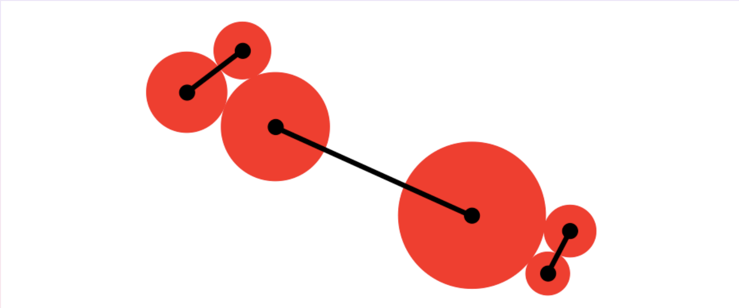
Grow a disk packing . . .



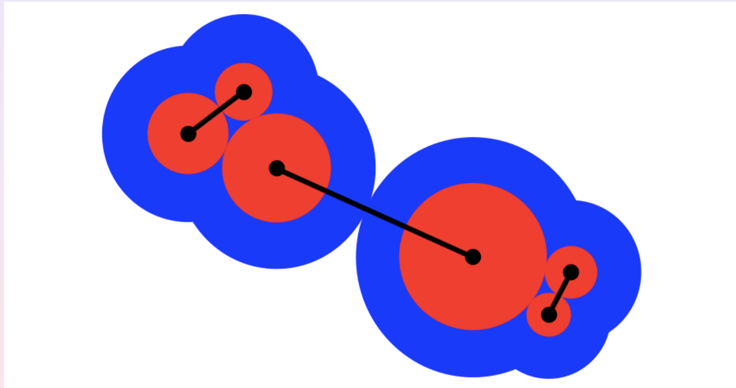
Maximize sum of radii.



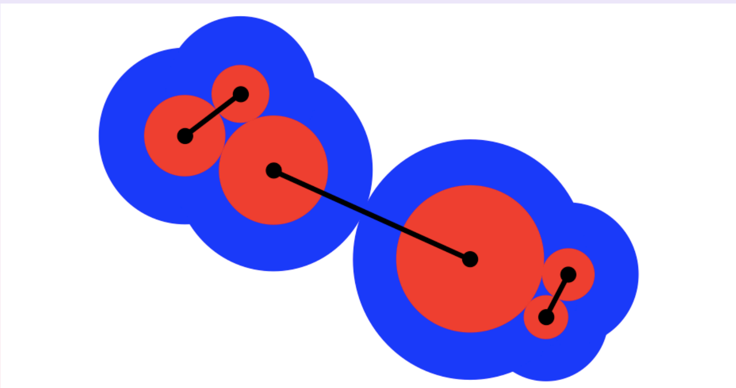
Doesn't work here . . .



Introduce moats . . .



Jack's results $\Longrightarrow$ This always works!



Jack's results $\Longrightarrow$ This always works!

Primal LP

$$\text{maximize} \quad \sum_{p\in P} r_p + \sum_{S\subset P,\ |S| \text{ odd and } 3\le |S|\le \frac{n}{2}} w_S$$

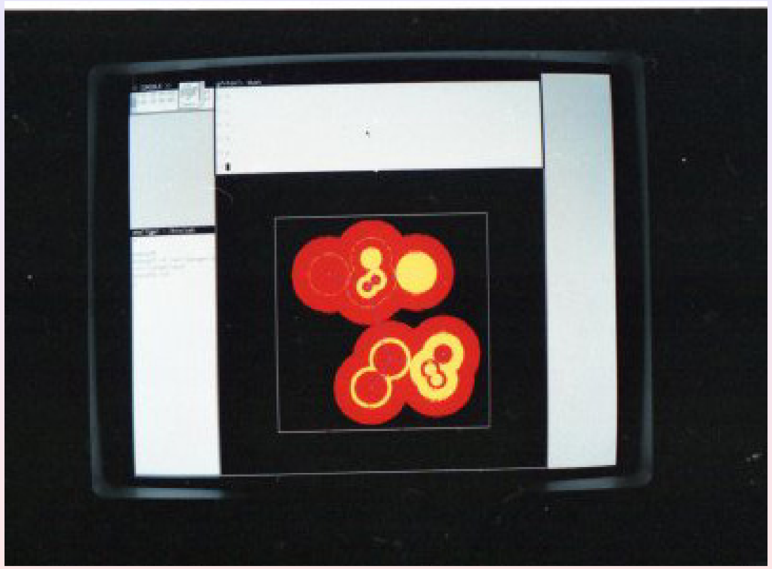
$$\begin{array}{rcll} r_p + r_q + \displaystyle\sum_{|S\cap\{p,q\}|=1} w_S & \le & d_{pq} & \text{for all } p,q\in P,\ q\ne p, \\ r_p & \ge & 0 & \text{for all } p\in P, \\ w_S & \ge & 0 & \text{for all } S\subset P,\ |S| \text{ odd and } 3\le |S| \le \dfrac{n}{2}. \end{array}$$

Dual LP

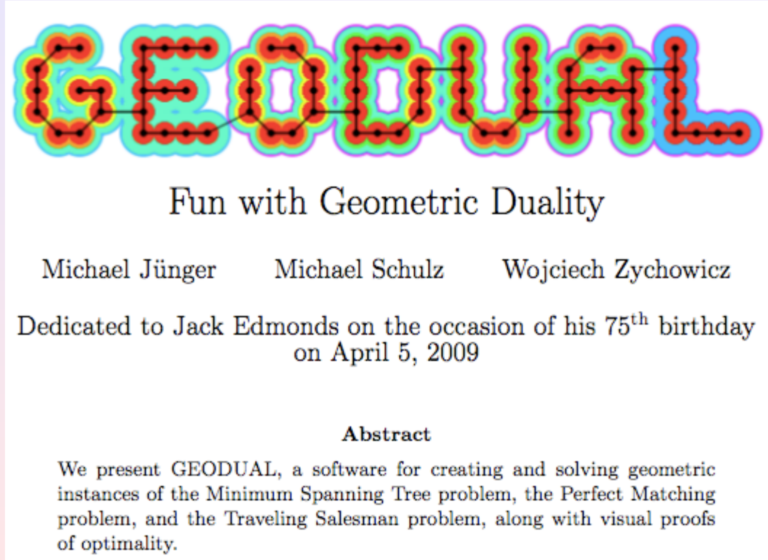
$$\text{minimize} \quad \sum_{p,q\in P,\ q\ne p} d_{pq}x_{pq}$$

$$\begin{array}{rcll} \displaystyle\sum_{q\in P,\ q\ne p} x_{pq} & \ge & 1 & \text{for all } p\in P, \\ \displaystyle\sum_{|S\cap\{p,q\}|=1} x_{pq} & \ge & 1 & \text{for all } S\subset P,\ |S| \text{ odd and } 3\le |S| \le \dfrac{n}{2}, \\ x_{pq} & \ge & 0 & \text{for all } p,q\in P,\ q\ne p. \end{array}$$

We (and others) had software in the early nineties.



This is brand new:



GEODUAL

Fun with Geometric Duality

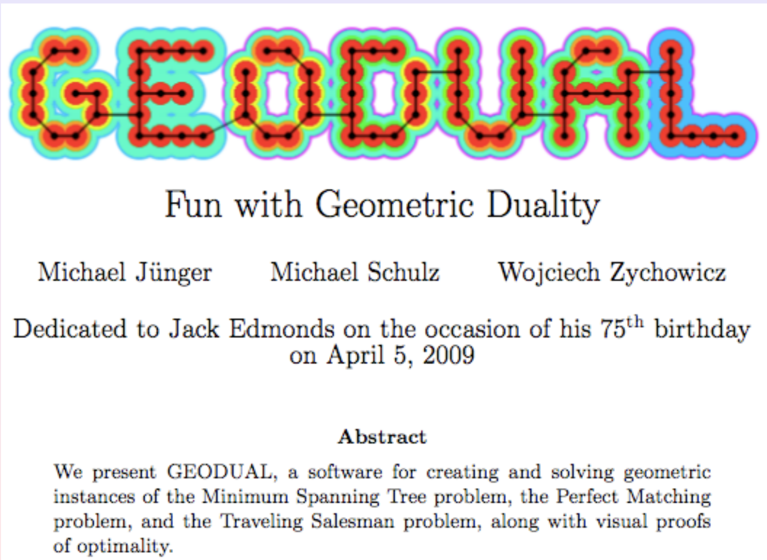
Michael Jünger Michael Schulz Wojciech Zychowicz

Dedicated to Jack Edmonds on the occasion of his 75$^{\text{th}}$ birthday on April 5, 2009

Abstract

We present GEODUAL, a software for creating and solving geometric instances of the Minimum Spanning Tree problem, the Perfect Matching problem, and the Traveling Salesman problem, along with visual proofs of optimality.

. . . and I'll give you a software demo now.



GEODUAL

Fun with Geometric Duality

Michael Jünger Michael Schulz Wojciech Zychowicz

Dedicated to Jack Edmonds on the occasion of his 75$^{\text{th}}$ birthday on April 5, 2009

Abstract

We present GEODUAL, a software for creating and solving geometric instances of the Minimum Spanning Tree problem, the Perfect Matching problem, and the Traveling Salesman problem, along with visual proofs of optimality.