Partial Metrics, Quasi-metrics and Oriented Hypercubes

Michel Deza

Ecole Normale Superieure, Paris, and JAIST, Ishikawa

Overview

Quasi-semi-metrics

Given a set X, a function $q : X \times X \to \mathbb{R}_{\geq 0}$ with $q(x,x)=0$ is a **quasi-distance** (or, in Topology, **prametric**) on X.

- A quasi-distance q is a **quasi-semi-metric** if for $x, y, z \in X$ it holds (**oriented triangle inequality**)

$$q(x,y) \leq q(x,z) + q(z,y)$$

.

- q' given by $q'(x,y)=q(y,x)$ is **dual** quasi-semi-metric to q.
- (X,q) can be partially ordered by the **specialization order**:

$$x \preceq y \text{ if and only if } q(x,y)=0.$$

Discrete quasi-metric on poset $(X, \leq)$ is $q_{\leq}(x,y)=0$ if $x \preceq y$ and $=1$ else; for $(X, q_{\leq})$, order $\preceq$ coincides with $\leq$.

Quasi-semi-metrics

Given a set X, a function $q : X \times X \rightarrow \mathbb{R}_{\geq 0}$ with $q(x,x)=0$ is a **quasi-distance** (or, in Topology, **prametric**) on X.

- A quasi-distance q is a **quasi-semi-metric** if for $x, y, z \in X$ it holds (**oriented triangle inequality**)

$$q(x,y) \leq q(x,z) + q(z,y)$$

.

- q' given by $q'(x,y)=q(y,x)$ is **dual** quasi-semi-metric to q.
- (X, q) can be partially ordered by the **specialization order**:

$$x \preceq y \text{ if and only if } q(x,y)=0.$$

  Discrete quasi-metric on poset $(X, \leq)$ is $q_{\leq}(x,y)=0$ if $x \preceq y$ and $=1$ else; for $(X, q_{\leq})$, order $\preceq$ coincides with $\leq$.
- A **weak quasi-metric** is a quasi-semi-metric q with **weak symmetry**: $q(x,y) = q(y,x)$ whenever $q(y,x) = 0$.
- An **Albert quasi-metric** is a quasi-semi-metric q with **weak definiteness**: $x = y$ whenever $q(x,y) = q(y,x) = 0$.

Quasi-metrics

A **quasi-metric** (or asymmetric, directed, oriented metric) is a quasi-semi-metric q with **definiteness**: $x = y$ iff $q(x, y) = 0$.

A **quasi-metric space** (X, q) is a set X with a quasi-metric q.

Asymmetric distances were introduced by Hausdorff in 1914.

Real world examples: one-way streets milage, travel time, transportation costs (up/downhill or up/downstream).

Quasi-metrics

A **quasi-metric** (or asymmetric, directed, oriented metric) is a quasi-semi-metric q with **definiteness**: $x = y$ iff $q(x, y) = 0$.
A **quasi-metric space** (X, q) is a set X with a quasi-metric q.
Asymmetric distances were introduced by Hausdorff in 1914.
Real world examples: one-way streets milage, travel time, transportation costs (up/downhill or up/downstream).

A quasi-metric q is **non-Archimedean** (or **quasi-ultrametric**) if it satisfy strengthened oriented triangle inequality

$$q(x, y) \leq \max\{q(x, z), q(z, y)\} \text{ for all } x, y, z \in X.$$

Cf. symmetric: distance, semi-metric, metric, ultrametric.

For a quasi-metric q, the functions $\frac{(q^p(x,y)+q^p(y,x))^{\frac{1}{p}}}{2}$, $p \geq 1$, (usually, $p = 1$ and $\frac{q(x,y)+q(y,x)}{2}$ is called **symmetrization** of q), $\max\{q(x, y), q(y, x)\}$, $\min\{q(x, y), q(y, x)\}$ are **metrics**.

Example: gauge quasi-metric

Given a compact convex region $B \subset \mathbb{R}^n$ containing origin, the **convex distance function** (or **Minkowski distance function**, **gauge**) is the quasi-metric on $\mathbb{R}^n$ defined, for $x \neq y$, by

$$q_B(x, y) = \inf\{\alpha > 0 : y - x \in \alpha B\}.$$

Equivalently, it is $\frac{||y-x||_2}{||z-x||_2}$, where z is unique point of the boundary $\partial(x + B)$ hit by the ray from x via y.

It holds $B{=}\{x \in \mathbb{R}^n : q_B(0, x) \leq 1\}$ with equality only for $x \in \partial B$.

If B is centrally-symmetric with respect to the origin, then q_B is a **Minkowskian metric** whose unit ball is B.

Examples: quasi-metrics on $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{S}^1$

- **Sorgenfrey quasi-metric** is a quasi-metric $q(x,y)$ on $\mathbb{R}$, equal to $y-x$ if $y \geq x$ and equal to 1, otherwise.
- Some similar quasi-metrics on $\mathbb{R}$ are:
  $q_1(x,y) = \max\{y-x, 0\}$ (l_1 **quasi-metric**),
  $q_2(x,y) = \min\{y-x, 1\}$ if $y \geq x$ and equal to 1, else,
  Given $a > 0$, $q_3(x,y) = y-x$ if $y \geq x$ and $=a(x-y)$, else.
  $q_4(x,y) = e^y - e^x$ if $y \geq x$ and equal to $e^{-y} - e^{-x}$, else.
- The **real half-line quasi-semi-metric** on $\mathbb{R}_{>0}$ is $\max\{0, \ln \frac{y}{x}\}$.
- The **circular-railroad quasi-metric** is a quasi-metric on the **unit circle** $\mathbb{S}^1 \subset \mathbb{R}^2$, defined, for any $x, y \in \mathbb{S}^1$, as the length of counter-clockwise circular arc from x to y in $\mathbb{S}^1$.

Digression: quasi-metrizable spaces

A topological space (X, τ) is called **quasi-metrizable space** if X admits a quasi-metric q such that the set of open q-balls $\{B(x, r) : r > 0\}$ form a neighborhood base at each $x \in X$.

More general **γ-space** is a topological space admitting a **γ-metric** q (a function $q : X \times X \rightarrow \mathbb{R}_{\geq 0}$ with $q(x, z_n) \rightarrow 0$ if $q(x, y_n) \rightarrow 0$ and $q(y_n, z_n) \rightarrow 0$) such that the set of open **forward** q-balls $\{B(x, r) : r > 0\}$ form a base at each $x \in X$.

Digression: quasi-metrizable spaces

A topological space (X, τ) is called **quasi-metrizable space** if X admits a quasi-metric q such that the set of open q-balls $\{B(x, r) : r > 0\}$ form a neighborhood base at each $x \in X$.

More general **γ-space** is a topological space admitting a **γ-metric** q (a function $q : X \times X \to \mathbb{R}_{\geq 0}$ with $q(x, z_n) \to 0$ if $q(x, y_n) \to 0$ and $q(y_n, z_n) \to 0$) such that the set of open **forward** q-balls $\{B(x, r) : r > 0\}$ form a base at each $x \in X$.

The **Sorgenfrey line** is the topological space $(\mathbb{R}, \tau)$ defined by the base $\{[a, b) : a, b \in \mathbb{R}, a < b\}$. It is not metrizable, 1st (not 2nd) countable paracompact (not locally compact) T_5**-space**.

But it is quasi-metrizable by **Sorgenfrey quasi-metric**:
$q(x, y) = y - x$ if $y \geq x$, and $q(x, y) = 1$, otherwise.

Digraph quasi-metric and metrics

- A **directed graph** (or **digraph**) is a pair $G = (V, A)$, where V is a set of vertices and A is a set of arcs.
- The **path quasi-metric** q_{dpath} in digraph $G{=}(V, A)$ is, for any $u, v \in V$, the length of a shortest $(u - v)$ path in G.
  Example: **Web hyperlink quasi-metric** (or **click count**) is q_{dpath} between two web pages (vertices of Web digraph).
- The **circular metric** (in digraph) is $q_{dpath}(u, v) + q_{dpath}(v, u)$.

Digraph quasi-metric and metrics

- A **directed graph** (or **digraph**) is a pair $G = (V, A)$, where V is a set of vertices and A is a set of arcs.
- The **path quasi-metric** q_{dpath} in digraph $G{=}(V, A)$ is, for any $u, v \in V$, the length of a shortest $(u - v)$ path in G.
  Example: **Web hyperlink quasi-metric** (or **click count**) is q_{dpath} between two web pages (vertices of Web digraph).
- The **circular metric** (in digraph) is $q_{dpath}(u, v) + q_{dpath}(v, u)$.
- Chartrand-Erwin-Raines-Zhang, 1999: the **strong metric** between $u, v \in V$ is the minimum number of edges of strongly connected subdigraph of G containing u and v.
- Chartrand-Erwin-Raines-Zhang, 2001: the **orientation metric** between 2 orientations D and D' of a graph is the minimum number of arcs of D whose directions must be reversed to produce an orientation isomorphic to D'.

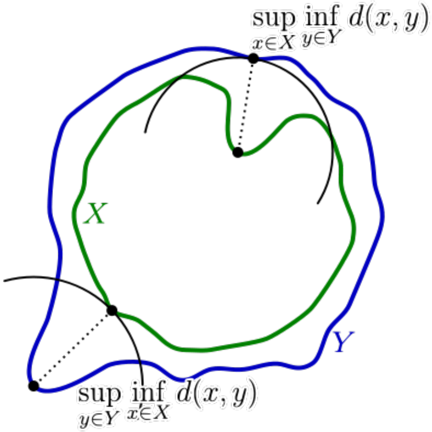
Examples at large

- In Psychophysics, the **probability-distance hypothesis**: the probability with which one stimulus is discriminated from another is a (continuously increasing) function of some subjective quasi-metric between these stimuli.
- Østvang, 2001, proposed a quasi-metric framework for relativistic gravity.
- The **Thurston quasi-metric** on the **Teichmüller space** T_g is $\frac{1}{2}\inf_h \ln ||h||_{Lip}$ for any $R_1^*, R_2^* \in T_g$, where $h : R_1 \rightarrow_2$ is a quasi-conformal homeomorphism, homotopic to the identity, and $||.||_{Lip}$ is the **Lipschitz norm** on the set of all injective functions $f : X \rightarrow Y$ defined by $||f||_{Lip} = \sup_{x,y \in X, x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x,y)}$.

Point-set distance and its applications

- In a (quasi)-metric space (X, d), the **point-set distance** between $x \in X$ and $A \subset X$ is $d(x, A) = \inf_{y \in A} d(x, y)$, The function $f_A(x) = d(x, A)$ is **distance map**. Distance maps are used in MRI (A is gray/white matter interface) as cortical maps, in Image Processing (A is image boundary), in Robot Motion (A is obstacle points set).
- $A \subset X$ is **Chebyshev set** if for each $x \in X$, there is unique **element of best approximation**:
$$y \in A \text{ with } d(x, y) = d(x, A).$$
If $A \subset X$ (usually, A is the boundary of a solid $X \subset \mathbb{R}^3$), **skeleton** of X is $\{x \in X : |\{y \in A : d(x, y) = d(x, A)\}| > 1\}$, i,e. all boundary points of **Voronoi regions** of points of A.
- The **directed Hausdorff distance** (on compact subspaces of (X, d)) is $q_{dHaus}(B, A) = \sup_{x \in B} d(x, A)$. The **Hausdorff metric** is $d_{Haus}(A, B) = \max\{q_{dHaus}(A, B), q_{dHaus}(B, A)\}$.

Hausdorff distance



http://en.wikipedia.org/wiki/User:Rocchini

A generalization: approach space

An **approach space** (Lowe, 1989) is a pair (X, D), where X is a set, and D is a **point-set function**, i.e., a function $D : X \times P(X) \to [0, \infty]$ (where $P(X)$ is the set of subsets of X) satisfying, for all $x \in X$ and all $A, B \subset X$, to:

1. $D(x, \{x\}) = 0$;
2. $D(x, \{\emptyset\}) = \infty$;
3. $D(x, A \cup B) = \min\{D(x, A), D(x, B)\}$;
4. $D(x, A) \leq D(x, A^{\epsilon}) + \epsilon$, for any $\epsilon \geq 0$ (here $A^{\epsilon} = \{x : D(x, A) \leq \epsilon\}$ is "ϵ-ball" with the center x).

Any **quasi-semi-metric space** (X, q) is an approach space with $D(x, A) = \min_{y \in A} q(x, y)$ (usual point-set distance).

Weightable quasi-semi-metrics

- A **weightable quasi-semi-metric** is a q-s-metric q on X admitting a weight function $w(x) \in \mathbb{R}$ on X with $q(x,y) - q(y,x) = w(y) - w(x)$ for all $x, y \in X$, i.e., $q(x,y) + \frac{1}{2}(w(x) - w(y))$ is its **symmetrization semi-metric** $\frac{q(x,y)+q(y,x)}{2}$.
- $w(x) + C$ is also such weight function for any constant C. If the set $\{q(x,y_0) - q(y_0,x)\}$ is bounded, then weight can be non-negative; then call $w'(x) = w(x) - \min_{y \in X} w(y) \geq 0$ **normalized weight function**.
- q is weightable iff $q(x,y) + w(x)$ is **partial semi-metric**.
- **Example**. Let q be quasi-metric on $X = V_3 = \{1,2,3\}$ with $q_{21} = q_{23} = 2$ and $q_{ij} = 1$ for other $1 \leq i \neq j \leq 3$. Then q is weightable with weight $w(i){=}1,0,1$ for $i{=}1,2,3$.

Partial semi-metrics

A function $p : X \times X \to \mathbb{R}_{\geq 0}$ with $p(x,y) = p(y,x)$ is a **partial semi-metric** (Matthews, 1992) if for $x, y, z \in X$, it holds
1) $p(x,x) \leq p(x,y)$ and
2) sharp triangle inequality:

$$p(x,y) \leq p(x,z) + p(z,y) - p(z,z).$$

Dropping **1)**: **weak partial semi-metric**. Example: $(\mathbb{R}_{\geq 0}, x+y)$.
If, moreover, **2)** is weakened to $p(x,y) \leq p(x,z)+p(z,y)$, then p is a **dislocated metric** (or Matthews **metric domain**).

Partial semi-metrics

A function $p : X \times X \to \mathbb{R}_{\geq 0}$ with $p(x, y) = p(y, x)$ is a **partial semi-metric** (Matthews, 1992) if for $x, y, z \in X$, it holds
1) $p(x, x) \leq p(x, y)$ and
2) sharp triangle inequality:

$$p(x, y) \leq p(x, z) + p(z, y) - p(z, z).$$

Dropping **1)**: **weak partial semi-metric**. Example: $(\mathbb{R}_{\geq 0}, x+y)$. If, moreover, **2)** is weakened to $p(x, y) \leq p(x, z)+p(z, y)$, then p is a **dislocated metric** (or Matthews **metric domain**).

Function p is a partial semi-metric iff q= $p(x, y)$-$p(x, x)$ is a **weightable q-s-metric** with $w(x)$=$p(x, x)$ and p is **partial metric** (i.e. T_0-separation holds: x=y if $p(x, x)$=$p(x, y)$=$p(y, y)$=0) if and only if, moreover, q is an **Albert quasi-metric**.

Güldürek and Richmond, 2005: every topology on a finite set X is defined, for $x \in X$, by $cl\{x\}$=$\{y \in X : y \preceq x\}$, where $x \preceq y$ means $p(x, y)$=$p(x, x)$ for a partial semi-metric p.

Weak partial semi-metrics

A function $p : X \times X \to \mathbb{R}_{\geq 0}$ with $p(x, y) = p(y, x)$ is a **weak partial semi-metric** (Heckmann, 1997) if for all $x, y, z \in X$, it holds $p(x, y) \leq p(x, z)+p(z, y)$-$p(z, z)$. For $x{=}y$, it gives the weakening $p(x, z) \geq \frac{p(x,x)+p(z,z)}{2}$ of $p(x, z) \geq p(x, x)$.

On any set X, $d(x, y){=}p(x, y)$-$\frac{p(x,x)+p(y,y)}{2}$, $w(x){=}\frac{p(x,x)}{2}$ and $p(x, y){=}d(x, y){+}w(x){+}w(y)$ is a bijection between weak partial semi-metrics p and weighted semi-metrics (d, w) $(w : X \to \mathbb{R}_{\geq 0})$. Moreover, p is partial metric iff d is metric.

Weak partial semi-metrics

A function $p: X \times X \to \mathbb{R}_{\geq 0}$ with $p(x,y) = p(y,x)$ is a **weak partial semi-metric** (Heckmann, 1997) if for all $x, y, z \in X$, it holds $p(x,y) \leq p(x,z)+p(z,y)$-$p(z,z)$. For x=y, it gives the weakening $p(x,z) \geq \frac{p(x,x)+p(z,z)}{2}$ of $p(x,z) \geq p(x,x)$.

On any set X, $d(x,y)$=$p(x,y)$-$\frac{p(x,x)+p(y,y)}{2}$, $w(x)$=$\frac{p(x,x)}{2}$ and $p(x,y)$=$d(x,y)$+$w(x)$+$w(y)$ is a bijection between weak partial semi-metrics p and weighted semi-metrics (d,w) $(w: X \to \mathbb{R}_{\geq 0})$. Moreover, p is partial metric iff d is metric.

In weak partial semi-metric space (X,p), define **open ball** $B(x,r)$=$\{y \in X : p(x,y) < r\}$. Call $U \subset X$ **open** if for all $x \in U$ there is $\epsilon > 0$ with $B(x,\epsilon) \subset U$. The open sets form topology with basis the balls $B(x,r)$; in general, not T_2 (Hausdorff).
Its **specialization preorder** induced by p is $x \preceq y$ if and only if $p(x,y) \leq p(a,a)$. It is partial order iff p is weak partial metric.

Digression on Semantics of Computation

A poset $(X, x \preceq y)$ is **dcpo** if it has a smallest element and each **directed subset** $A \subset X$ (i.e. $A \neq \emptyset$ and for any $x, y \in A$, exists $z \in A$ with $x, y \preceq z$) has a supremum $\sup A$ in X.
Let X^C be the set of **compact** $x \in X$, i.e. for each directed subset A with $x \preceq \sup A$, there is $a \in A$ with $x \preceq a$.
A **Scott domain** is a dcpo where all sets $\{a \in X^C : a \preceq x\}$ are directed with sup$=x$ and each **consistent** $A \subset X$ (i.e. there exists $x \in X$ with $a \preceq x$ for all $a \in A$) has supremum in X.
Main examples: all words over finite alphabet with prefix order, all *vague real numbers* (nonempty segments of $\mathbb{R}$) with reverse inclusion order, all subsets of $\mathbb{N}$ under inclusion

Digression on Semantics of Computation

A poset $(X, x \preceq y)$ is **dcpo** if it has a smallest element and each **directed subset** $A \subset X$ (i.e. $A \neq \emptyset$ and for any $x, y \in A$, exists $z \in A$ with $x, y \preceq z$) has a supremum $\sup A$ in X.

Let X^C be the set of **compact** $x \in X$, i.e. for each directed subset A with $x \preceq \sup A$, there is $a \in A$ with $x \preceq a$.

A **Scott domain** is a dcpo where all sets $\{a \in X^C : a \preceq x\}$ are directed with sup$=x$ and each **consistent** $A \subset X$ (i.e. there exists $x \in X$ with $a \preceq x$ for all $a \in A$) has supremum in X.

Main examples: all words over finite alphabet with prefix order, all *vague real numbers* (nonempty segments of $\mathbb{R}$) with reverse inclusion order, all subsets of $\mathbb{N}$ under inclusion

Quantitative Domain Theory: a "distance" between programs (points of a semantic domain) is used to quantify speed (of processing or convergence) or complexity of programs.

$x \preceq y$ (program y contains all info from x) is **specialization preorder** ($x \preceq y$ iff $p(x, y) = p(x, x)$) for a **partial metric** p on X.

Quantale-valued partial metrics

Scott's domain theory gave partial order and non-Hausdorff topology on partial objects in computation.
In computation over a metric space of totally defined objects, partial metric models partially defined information: $p(x,x) > 0$ ($=0$) mean that object x is partially (**totally**) defined.

A **quantale** is a complete lattice M with an associative binary operation $*$ with $x * \vee_{i \in I} y_i = \vee_{i \in I}(x * y_i)$, $\vee_{i \in I} y_i * x = \vee_{i \in I}(y_i * x)$.
Kooperman-Mattews-Rajoonesh, 2004: any topology can arise from a quantale-valued partial metric.

Quantale-valued partial metrics

Scott's domain theory gave partial order and non-Hausdorff topology on partial objects in computation.
In computation over a metric space of totally defined objects, partial metric models partially defined information: $p(x,x) > 0$ ($=0$) mean that object x is partially (**totally**) defined.

A **quantale** is a complete lattice M with an associative binary operation $*$ with $x * \vee_{i \in I} y_i = \vee_{i \in I}(x * y_i)$, $\vee_{i \in I} y_i * x = \vee_{i \in I}(y_i * x)$.
Kooperman-Mattews-Rajoonesh, 2004: any topology can arise from a quantale-valued partial metric.

Another way to see: fuzzy non-reflexive equalities. Hohle, 1992: for a commutative quantale $M = (M, \leq, 1, 0, \vee, \wedge, *)$, **multivalued** (**$M$-valued**) set is a set X equipped with a **fuzzy equality**, i.e., a map $E : X \times X \to M$ subject to $E(x,x) = 1$, $E(x,y) = E(y,x)$ and $E(x,y) * E(y,z) \leq E(x,z)$ for $x, y, z \in X$.

$WQSMET_n$ and $PSMET_n$, $wPSMET_n$

Clearly, all weightable quasi-semi-metrics on n-set $X = [n] = \{1, 2, \dots, n\}$ form a polyhedral convex cone of dimension $\binom{n}{2} + n = \binom{n+1}{2}$. Denote it by $WQSMET_n$.
$WQSMET_n$ is the section of $QSMET_n$ by $\binom{n}{3}$ hyperplanes $xyzx = xzyx$ of **relaxed symmetry** defined next.

Denote by $PSMET_n$ and $wPSMET_n$ the cones of partial and weak partial semi-metrics on n-points.
They have $3\binom{n}{3}+n^2$ and $3\binom{n}{3}+\binom{n+1}{2}$ facets, respectively. They are relaxations of $\binom{n}{2}$-dim. cone $SMET_n$ of all n-points semi-metrics.

Relaxed and cyclic symmetry

- Quasi-semi-metric q on X has **relaxed symmetry** ($xyzx = xzyx$) if for different $x, y, z \in X$ it holds
  $q(x,y) + q(y,z) + q(z,x) = q(x,z) + q(z,y) + q(y,x)$, i.e.
  $q(x,y) - q(y,x) = (q(z,y) - q(y,z)) - (q(z,x) - q(x,z))$,
  Equivalently, q is **weightable**: fix point z_0 and define
  $w(x) = q(z_0,x) - q(x,z_0)$.

Relaxed and cyclic symmetry

- Quasi-semi-metric q on X has **relaxed symmetry** ($xyzx = xzyx$) if for different $x, y, z \in X$ it holds
  $q(x,y) + q(y,z) + q(z,x) = q(x,z) + q(z,y) + q(y,x)$, i.e.
  $q(x,y) - q(y,x) = (q(z,y) - q(y,z)) - (q(z,x) - q(x,z))$,
  Equivalently, q is **weightable**: fix point z_0 and define
  $w(x) = q(z_0, x) - q(x, z_0)$.
- Given $k \geq 3$, quasi-semi-metric q is **k-cyclically symmetric** if
  $x_1x_2x_3 \dots x_kx_1 = x_1x_kx_{k-1} \dots x_2x_1$, for $x_1x_2 \dots x_k \in X$.
  The case $k = 3$ (relaxed symmetry) is equivalent to the general case of any $k \geq 3$. For example, for $k = 4$,
  $(x_1x_2x_3x_1\text{-}x_1x_3x_2x_1)+(x_1x_3x_4x_1\text{-}x_1x_4x_3x_1)=$
  $x_1x_2x_3x_4x_1\text{-}x_1x_4x_3x_2x_1$ and, in other direction,
  $(x_1x_2x_3x_4x_1\text{-}x_1x_4x_3x_2x_1)+(x_1x_2x_4x_3x_1\text{-}x_1x_3x_4x_2x_1)+$
  $(x_1x_4x_2x_3x_1\text{-}x_1x_3x_2x_4x_1)=2\ (x_1x_2x_3x_1\text{-}x_1x_3x_2x_1)$.

Realizations by weighted (di)graphs

- Any finite semi-metric d is the shortest path semi-metric of a $\mathbb{R}_{\geq 0}$-weighted graph G.
  G can be a tree if and only if d satisfy to **4-points inequality**:
  $d(x,y) + d(z,u) \leq \max\{d(x,z) + d(y,u), d(x,u) + d(y,z)\}$.

Realizations by weighted (di)graphs

- Any finite semi-metric d is the shortest path semi-metric of a $\mathbb{R}_{\geq 0}$-weighted graph G.
  G can be a tree if and only if d satisfy to **4-points inequality**:
  $d(x,y)+d(z,u) \leq \max\{d(x,z)+d(y,u), d(x,u)+d(y,z)\}$.
- Any finite quasi-semi-metric q is the shortest path q-s-metric of a $\mathbb{R}_{\geq 0}$-weighted digraph G.
  Patrinos-Hakimi, 1972: G can be a **bidirectional tree** (a tree with all edges replaced by 2 oppositely directed arcs) if and only if q is **weightable** and $q(x,y)+q(y,x)$ is tree-realizable.

Weightable hitting time quasi-metric

Given connected graph $G = (V, E)$ with $|E| = m$, consider random walks on G, where at each step walk moves with uniform probability from current vertex a neighboring one.

The **hitting time quasi-metric** $H(u, v)$ from $u \in V$ to $v \in V$ is the expected number of steps (edges) for a random walk on G beginning at u to reach v for the first time; put $H(u, u) = 0$. This quasi-metric is **weightable**.

Weightable hitting time quasi-metric

Given connected graph $G = (V, E)$ with $|E| = m$, consider random walks on G, where at each step walk moves with uniform probability from current vertex a neighboring one.

The **hitting time quasi-metric** $H(u, v)$ from $u \in V$ to $v \in V$ is the expected number of steps (edges) for a random walk on G beginning at u to reach v for the first time; put $H(u, u) = 0$. This quasi-metric is **weightable**.

The **commuting time metric** is $C(u, v) = H(u, v) + H(v, u)$. It holds $C((u, v) = 2m\Omega(u, v)$, where $\Omega(u, v)$ is the **effective resistance metric**: 0 if $u = v$ and, else, $\frac{1}{\Omega(u,v)}$ is the current flowing into grounded v when potential 1 volt is applied to u (each edge is seen as a resistor of 1 ohm). $\Omega(u, v)$ is $\sup_{f: V \to \mathbb{R},\, D(f)>0} \frac{(f(u)-f(v))^2}{D(f)}$ with $D(f) = \sum_{st \in E}(f(s) - f(t))^2$.

z_0-derivations of semi-metrics

Given semi-metric space (X, d) and $z_0 \in X$, its **z_0-derivation** is q-s-metric $q(x, y)=\frac{1}{2}(d(x, y)+d(y, z_0)$-$d(x, z_0))$. So, $d=q + q'$, q is weightable with $w(x)=d(x, z_0)=q(z_0, x)$ and $q(x, z_0)=0$. Weightable q-s-metric q is z_0-derivation of $q+q'$ iff $q(x, z_0)=0$.

Quasi-metric q is **z_0-derivation** of a metric d iff **partial metric** $p(x, y)=q(x, y) + w(x))$ is $\frac{1}{2}(d(x, y)+d(y, z_0)+d(x, z_0))$.

Clearly, z_0-derivations of semi-metrics $d \in SMET_n$ for fixed $z_0 = i \in X = [n]$ form a cone $\mathbf{D_iWQSMET_n} \subset WQSMET_n$.

Any inequality $\sum_{1 \le i,j \le n} a_{ij} dij \ge 0$, valid for $d \in SMET_n$, implies, valid for $q \in D_{z_0} WQSMET_n$, inequality
$\sum_{1 \le i,j \le n} a_{ij} qij + \sum_{1 \le i,j \le n} a_{ij} d(j, z_0) - \sum_{1 \le i,j \le n} a_{ij} d(i, z_0) \ge 0$.

l_p-quasi-metrics

- On a **normed vector space** $(V, ||.||)$, its **norm metric** is
$$||x-y||$$
.
The **l_p-metric** is $||x-y||_p$ norm metric on $\mathbb{R}^m$ (or on $\mathbb{C}^m$):
$||x||_p = (\sum_{i=1}^m |x_i|^p)^{\frac{1}{p}}$ for $p \geq 1$ and $||x||_\infty = \max_{1 \leq i \leq m} |x_i|$.
The **Euclidean metric** (or **Pythagorean distance**, **as-crow-flies distance**, **beeline distance**) is l_2-metric on $\mathbb{R}^m$.

l_p-quasi-metrics

- On a **normed vector space** $(V, ||.||)$, its **norm metric** is
$$||x - y||$$
.
The **l_p-metric** is $||x-y||_p$ norm metric on $\mathbb{R}^m$ (or on $\mathbb{C}^m$):
$||x||_p = (\sum_{i=1}^{m} |x_i|^p)^{\frac{1}{p}}$ for $p \geq 1$ and $||x||_\infty = \max_{1 \leq i \leq m} |x_i|$.
The **Euclidean metric** (or **Pythagorean distance**, **as-crow-flies distance**, **beeline distance**) is l_2-metric on $\mathbb{R}^m$.
- **l_p-quasi-metric** on $\mathbb{R}^m$ is z_0-derivation of l_p-metric with $z_0 = (0, \dots, 0)$, i.e. it is **oriented l_p-norm** $||x-y||_{p,or} =$
$(\sum_{i=1}^{m} |x_i - y_i|^p)^{\frac{1}{p}} + (\sum_{i=1}^{m} |y_i|^p)^{\frac{1}{p}} - (\sum_{i=1}^{m} |x_i|^p)^{\frac{1}{p}}$ and
$l^m_{p,or}$ is the quasi-metric space $(\mathbb{R}^m, ||x-y||_{p,or})$,
l_p-$QSMET_n$ is the set of all l_p q-s-metrics on n points; it is (as for semi-metrics) a cone exactly for $p = 1, \infty$.
- $(l_2\text{-}QSMET_n)^2 = \{q^2 : q \in l_2\text{-}QSMET_n\}$ is a cone also.

l_1 and l_∞ quasi-metrics

- In particular, **l_1-quasi-metric** on $\mathbb{R}^m_{\geq 0}$ is $\sum_{i=1}^m (|x_i - y_i| + |y_i| - |x_i|) = 2\sum_{i=1}^m \max\{y_i - x_i, 0\}$ and **l_∞-quasi-metric** is $2\max_{1\leq i\leq m} \max\{y_i - x_i, 0\}$.
- Any q-s-metric q on n points **embeds in $l^m_{1,or}$ for some m iff** $q \in OCUT_n$ (the cone generated by all oriented cuts on $[n]$).
- Any q-s-metric q on n points **embeds into $l^n_{\infty,or}$**.
  In fact, let $v_1, \dots, v_n \in \mathbb{R}^n$ be
  $v_i = (q(i,1), q(i,2), \dots, q(i,n))$.
  Then $||v_i - v_j||_{\infty,or} = max_k(q(j,k) - q(i,k), 0) \leq q(j,i)$,
  while $q(j,i) - q(i,i) = q(j,i)$; so, $||v_i - v_j||_{\infty,or} = q(j,i)$.

l_1 and l_∞ quasi-metrics

- In particular, **l_1-quasi-metric** on $\mathbb{R}^m_{\geq 0}$ is $\sum_{i=1}^m (|x_i - y_i| + |y_i| - |x_i|) = 2\sum_{i=1}^m \max\{y_i - x_i, 0\}$ and **l_∞-quasi-metric** is $2\max_{1\leq i\leq m}\max\{y_i - x_i, 0\}$.
- Any q-s-metric q on n points **embeds in $l^m_{1,or}$ for some m iff** $q \in OCUT_n$ (the cone generated by all oriented cuts on $[n]$).
- Any q-s-metric q on n points **embeds into $l^n_{\infty,or}$**.
  In fact, let $v_1, \dots, v_n \in \mathbb{R}^n$ be
  $v_i = (q(i,1), q(i,2), \dots, q(i,n))$.
  Then $||v_i - v_j||_{\infty,or} = max_k(q(j,k) - q(i,k), 0) \leq q(j,i)$,
  while $q(j,i) - q(i,i) = q(j,i)$; so, $||v_i - v_j||_{\infty,or} = q(j,i)$.

Example: on $\mathbb{R}_{\geq 0}$, to the partial metric $p(x,y) = \max\{x,y\}$ corresponds l_1 **quasi-metric** $q(x,y) = \max\{x,y\}$-$x = \max\{y$-$x, 0\}$ with weight $w(x) = x$ and $d(x,y) = \frac{q(x,y)+q(y,x)}{2} = \frac{|x-y|}{2} = p(x,y)$- $\frac{x+y}{2}$ (twice l_1 **metric**).

Embedding between l_p quasi-metrics

Clearly, any isometric embedding f of semi-metric spaces (X, d_X) into (Y, d_Y) is isometric embedding of z_0-derivations of (X, d_X) into $f(z_0)$-derivation of (Y, d_Y).
So (as well as for semi-metrics), it holds:

- Any l_p-quasi-metric with $1 \leq p \leq 2$ is a l_1-quasi-metric.
- Any l_1-quasi-metric is the square of a l_2-quasi-metric.
- Any quasi-metric is a l_∞-quasi-metric.

So, $l_2\text{-}QSMET_n \subset l_1\text{-}QSMET_n \subset (l_2\text{-}QSMET_n)^2$ holds; it generalizes $l_2\text{-}SMET_n \subset l_1\text{-}SMET_n \subset (l_2\text{-}SMET_n)^2$, where, for semi-metrics, $(l_2\text{-}SMET_n)^2$ is the **negative type cone** NEG_n and $l_1\text{-}SMET_n$ is the **cut cone** CUT_n.

Measure quasi-semi-metric versus l_1

- Given a **measure space** $(\Omega, \mathcal{A}, \mu)$, the **symmetric difference** (or **measure**) **semi-metric** on the set $\mathcal{A}_\mu = \{A \in \mathcal{A} : \mu(A) < \infty\}$ is $\mu(A \triangle B)$ (where $A \triangle B = (A \cup B) \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A)$ is the **symmetric difference** of sets A, B) and 0 if $\mu(A \triangle B) = 0$. Identifying $A, B \in \mathcal{A}_\mu$ if $\mu(A \triangle B) = 0$, gives the **measure metric**. If $\mu(A) = |A|$, then $\mu(A \triangle B) = |A \triangle B|$ is a metric.

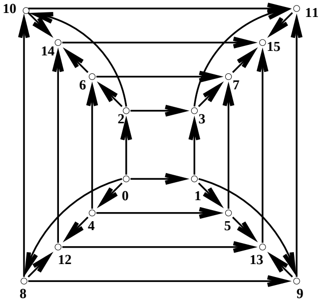
Measure quasi-semi-metric versus l_1

- Given a **measure space** $(\Omega, \mathcal{A}, \mu)$, the **symmetric difference** (or **measure**) **semi-metric** on the set $\mathcal{A}_\mu = \{A \in \mathcal{A} : \mu(A) < \infty\}$ is $\mu(A \triangle B)$ (where $A \triangle B = (A \cup B) \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A)$ is the **symmetric difference** of sets A, B) and 0 if $\mu(A \triangle B) = 0$. Identifying $A, B \in \mathcal{A}_\mu$ if $\mu(A \triangle B) = 0$, gives the **measure metric**. If $\mu(A) = |A|$, then $\mu(A \triangle B) = |A \triangle B|$ is a metric.
- **Measure quasi-semi-metric** on the set $\mathcal{A}_\mu$ is z_0-derivation of the measure semi-metric for $z_0 = \emptyset$, i.e. it is $q(A, B) = \mu(A \triangle B) + \mu(B) - \mu(A) = \mu(B \backslash A)$.

In fact (as well as in the metric case), a q-s-metric is l_1-quasi-metric **if and only if** it is a measure quasi-metric.

n-cube: inclusion (Boolean) orientation

Label vertices of n-cube by numbers $0, \dots, 2^n - 1$; their binary expansions label all subsets A of $[n] = \{1, \dots, n\}$.
Hasse diagram of the **Boolean lattice** $2^{[n]}$ is **inclusion-oriented n-cube**: do arc from A to B if $A \subset B$ and $|B \backslash A|=1$.
Its **path quasi-semi-metric** is $|B \backslash A|$ if $A \subset B$ and $=\infty$, else, while **Hamming semi-distance** is l_1 quasi-metric $|B \backslash A|$, i.e.
$|B \backslash (B \cap A)|=\sum_{i=1}^n \max\{1_{i \in B} - 1_{i \in A}, 0\}=\sum_{i=1}^n 1_{i \in B}(1 - 1_{i \in A})$.



The cones under consideration

$l_1 SMET_n = CUT_n = MCUT_n = BSMET_n \subset SMET_n = l_\infty SMET_n$;

$l_1 QSMET_n = OCUT_n \subset WQSMET_n \subset QSMET_n = l_\infty QSMET_n$,

and $OCUT_n \subset OMCUT_n \subset BQSMET_n \subset QSMET_n$, where

$MCUT_n$, $OMCUT_n$ are generated by multicuts, o-multicuts, and $BSMET_n$, $BQSMET_n$ are generated by $\{0,1\}$-valued semi-metrics, $\{0,1\}$-valued quasi-semi-metrics.

Also, l_1-$PSMET_n = BPSMET_n \subset PSMET_n$, where
$PSMET_n = \{p = ((p_{ij} = q_{ij} + w_i))\} : q = ((q_{ij})) \in WQSMET_n$,
l_1-$PSMET_n = \{p = ((p_{ij} = q_{ij} + w_i))\} : q = ((q_{ij})) \in OCUT_n$, and
$BPSMET_n$ is generated by $\{0,1\}$-valued $p \in PSMET_n$.

Oriented cut quasi-semi-metrics

Given a subset S of $[n] = \{1, \ldots, n\}$, the **oriented cut quasi-semi-metric** (or **o-cut**) $\delta(S)'$ is a quasi-semi-metric on $[n]$:

$$\delta'_{ij}(S) = |(S \cap \{i\}) \backslash (S \cap \{j\})| = \begin{cases} 1, & \text{if} \quad i \in S, j \notin S, \\ 0, & \text{otherwise.} \end{cases}$$

$\delta'(S)$ is, for any $z_0 \in \overline{S}$, **z_0-derivation** of the **cut semi-metric** $\delta(S)=\delta'(S) + \delta'([n] \backslash S)$ (twice of symmetrization of $\delta'(S)$).
Quasi-semi-metric $\delta'(S)$ is **weightable** with $w(i) = 1_{i \notin S}$.

Oriented cut quasi-semi-metrics

Given a subset S of $[n]=\{1,\dots,n\}$, the **oriented cut quasi-semi-metric** (or **o-cut**) $\delta(S)'$ is a quasi-semi-metric on $[n]$:

$$\delta'_{ij}(S)=|(S\cap\{i\})\setminus(S\cap\{j\})|=\begin{cases}1, & \text{if} \quad i\in S, j\notin S,\\ 0, & \text{otherwise.}\end{cases}$$

$\delta'(S)$ is, for any $z_0\in\overline{S}$, **z_0-derivation** of the **cut semi-metric** $\delta(S)=\delta'(S)+\delta'([n]\setminus S)$ (twice of symmetrization of $\delta'(S)$).
Quasi-semi-metric $\delta'(S)$ is **weightable** with $w(i)=1_{i\notin S}$.

Oriented cut cone $OCUT_n$ is $\binom{n+1}{2}$-dimensional subcone of $WQSMET_n$ generated by 2^n-2 non-zero o-cuts $\delta'(S)$ of $[n]$.

$OCUT_n$=l_1-$QSMET_n$, the cone of n points l_1 q-s-metrics.

Oriented multicut quasi-semi-metrics

Given an **ordered partition** $\{S_1, \dots, S_t\}$, $t \geq 2$, of $[n]$, **oriented multicut quasi-semi-metric** (or **o-multicut**) $\delta'(S_1, \dots, S_t)$ is:

$$\delta'_{ij}(S_1, \dots, S_t) = \begin{cases} 1, & \text{if} \quad i \in S_h, j \in S_m, m > h, \\ 0, & \text{otherwise.} \end{cases}$$

The **multicut semi-metric** $\delta(S_1, \dots, S_t)$ is symmetrization $\delta'(S_1, \dots, S_t) + \delta'(S_t, \dots, S_1)$ of q-s-metric $2\delta'(S_1, \dots, S_t)$.

Oriented multicut quasi-semi-metrics

Given an **ordered partition** $\{S_1, \dots, S_t\}$, $t \geq 2$, of $[n]$, **oriented multicut quasi-semi-metric** (or **o-multicut**) $\delta'(S_1, \dots, S_t)$ is:

$$\delta'_{ij}(S_1, \dots, S_t) = \begin{cases} 1, & \text{if} \quad i \in S_h, j \in S_m, m > h, \\ 0, & \text{otherwise.} \end{cases}$$

The **multicut semi-metric** $\delta(S_1, \dots, S_t)$ is symmetrization $\delta'(S_1, \dots, S_t) + \delta'(S_t, \dots, S_1)$ of q-s-metric $2\delta'(S_1, \dots, S_t)$.

An o-multicut $\delta'(S_1, S_2)$ is exactly o-cut $\delta'(S_1)$.

Lemma: o-cuts are exactly weightable o-multicut q-s-metrics

In fact, let $i \in S_1$, $j \in S_2$, $k \in S_3$ in q-s-metric $q = \delta'_{ij}(S_1, \dots, S_q)$. If q is weightable, then $q(i,j) = w(j) - w(i) = 1$. Impossible, since $q(i,k) = w(k) - w(i) = 1, q(j,k) = w(k) - w(j) = 1$.

Oriented cuts with $n = 3$

There are 7 oriented cut q-s-metrics on 3 points, given by binary $\binom{3}{2}$-vectors indexed as $(12, 13; 21, 23; 31, 32)$:

$$\begin{gathered}
\delta'(\{\emptyset\}) = \delta'(\{1,2,3\}) = (0,0;0,0;0,0),\\
\delta'(\{1\}) = (1,1;0,0;0,0),\\
\delta'(\{2\}) = (0,0;1,1;0,0),\\
\delta'(\{3\}) = (0,0;0,0;1,1),\\
\delta'(\{1,2\}) = (0,1;0,1;0,0),\\
\delta'(\{1,3\}) = (1,0;0,0;0,1),\\
\delta'(\{2,3\}) = (0,0,1,0,1,0).
\end{gathered}$$

Oriented cuts with $n = 3$

There are 7 oriented cut q-s-metrics on 3 points, given by binary $\binom{3}{2}$-vectors indexed as $(12, 13; 21, 23; 31, 32)$:

$$\begin{gathered}
\delta'(\{\emptyset\}) = \delta'(\{1,2,3\}) = (0,0;0,0;0,0), \\
\delta'(\{1\}) = (1,1;0,0;0,0), \\
\delta'(\{2\}) = (0,0;1,1;0,0), \\
\delta'(\{3\}) = (0,0;0,0;1,1), \\
\delta'(\{1,2\}) = (0,1;0,1;0,0), \\
\delta'(\{1,3\}) = (1,0;0,0;0,1), \\
\delta'(\{2,3\}) = (0,0,1,0,1,0).
\end{gathered}$$

Example. Let again q be quasi-metric on $X = V_3 = \{1,2,3\}$ with $q_{21} = q_{23} = 2$ and $q_{ij} = 1$ for other $1 \leq i \neq j \leq 3$.
Then $q = \delta'(\{1\}) + 2\delta'(\{2\}) + \delta'(\{3\})$, i.e. $q \in OCUT_3$.

Oriented multicuts versus oriented cuts

There are 6 oriented multicuts on 3 points, in addition to 7 oriented cuts, listed above:

$$\begin{aligned}
\delta'(\{1\},\{2\},\{3\}) &= (1,1;0,1;0,0),\\
\delta'(\{2\},\{1\},\{3\}) &= (0,1;1,0;0,0),\\
\delta'(\{1\},\{3\},\{2\}) &= (1,1;0,0;0,1),\\
\delta'(\{2\},\{3\},\{1\}) &= (0,0;1,1;1,0),\\
\delta'(\{3\},\{1\},\{2\}) &= (1,0;0,1;1,1),\\
\delta'(\{3\},\{2\},\{1\}) &= (0,0;1,0;1,1).
\end{aligned}$$

Oriented multicuts versus oriented cuts

There are 6 oriented multicuts on 3 points, in addition to 7 oriented cuts, listed above:

$$\begin{aligned}
\delta'(\{1\},\{2\},\{3\}) &= (1,1;0,1;0,0),\\
\delta'(\{2\},\{1\},\{3\}) &= (0,1;1,0;0,0),\\
\delta'(\{1\},\{3\},\{2\}) &= (1,1;0,0;0,1),\\
\delta'(\{2\},\{3\},\{1\}) &= (0,0;1,1;1,0),\\
\delta'(\{3\},\{1\},\{2\}) &= (1,0;0,1;1,1),\\
\delta'(\{3\},\{2\},\{1\}) &= (0,0;1,0;1,1).
\end{aligned}$$

Every **multicut** is $\mathbb{R}_{\geq 0}$-linear combination of cuts, while any **oriented multicut** with $t > 2$ is a $\mathbb{R}$-linear but not $\mathbb{R}_{\geq 0}$-linear combination of o-cuts, since it is non-weightable q-s-metric.

The number of oriented multicuts on $[n]$ is ordered Bell number $Bo(n)$ (the sequence A00670 in Sloan's OEIS).

Linear description of $QSMET_n$

cone	dim.	Nr. of ext. rays (orbits)	Nr. of facets (orbits)	diam.
$OMCUT_3$ $=QSMET_3$	6	12(2)	12(2)	2; 2
$OMCUT_4$	12	74(5)	72(4)	2; 2
$QSMET_4$	12	164(10)	36(2)	3; 2
$OMCUT_5$	20	540(9)	35320(194)	2; 3
$QSMET_5$	20	43590(229)	80(2)	3; 2
$OMCUT_6$	30	4682(19)	$> 2.1 \cdot 10^9 (> 1.6 \cdot 10^6)$	2; ?
$QSMET_6$	30	$> 1.8 \cdot 10^9 (> 1.2 \cdot 10^6)$	150(2)	?; 2

The orbits are under the symmetry group $Z_2 \times Sym(n)$: $n!$ permutations of $[n] = \{1, \dots, n\}$ and the reversal $(ij) \rightarrow (ji)$.

Linear description of $QSMET_n$

cone	dim.	Nr. of ext. rays (orbits)	Nr. of facets (orbits)	diam.
$OMCUT_3$ $=QSMET_3$	6	12(2)	12(2)	2; 2
$OMCUT_4$	12	74(5)	72(4)	2; 2
$QSMET_4$	12	164(10)	36(2)	3; 2
$OMCUT_5$	20	540(9)	35320(194)	2; 3
$QSMET_5$	20	43590(229)	80(2)	3; 2
$OMCUT_6$	30	4682(19)	$> 2.1 \cdot 10^9 (> 1.6 \cdot 10^6)$	2; ?
$QSMET_6$	30	$> 1.8 \cdot 10^9 (> 1.2 \cdot 10^6)$	150(2)	?; 2

The orbits are under the symmetry group $Z_2 \times Sym(n)$: $n!$ permutations of $[n] = \{1, \dots, n\}$ and the reversal $(ij) \to (ji)$.

$QSMET_n$ has $n(n-1)^2$ **facets** in 2 orbits: $6\binom{n}{3}$ oriented triangle inequalities and $n(n-1)$ inequalities $q(x, y) \geq 0$.
Moreover, they are also facets of $OCUT_n$ and so, of cones $WQSMET_n$, $OMCUT_n$ and $BQSMET_n$ containing $OCUT_n$.

Cones on 3 points (all 6-dimensional)

The cone $OCUT_3$ of l_1 q-s-metrics on 3 points **coincides** with the cone of weightable quasi-semi-metrics $WQSMET_3$.
It has 6 extreme rays in 2 orbits of sizes $3, 3$ represented by o-cuts $\delta^{'}(\{1\}){=}(1,1;0,0;0,0)$ and $\delta^{'}(\overline{\{1\}}){=}(0,0;1,0;1,0)$,
and $9 = 6+3$ facets represented by $q_{ij} \geq 0$ and $Tr_{ij,k} \geq 0$.

Larger cone $OMCUT_3 = BQSMET_3 = QSMET_3$ has 12 extreme rays in 3 orbits represented by two above o-cuts and the **o-multicut** $\delta^{'}(\{1\},\{2\},\{3\}) = (1,1;0,1;0,0)$,
and $12 = 6+6$ facets represented by $q_{ij} \geq 0$ and $Tr_{ij,k} \geq 0$.

Cone l_1-$PSMET_3{=}PSMET_3$ has 13=1+3+3+3+3 extreme rays represented by $(1,1;1,1;1,1)$, $P(\delta^{'}(\{1\}))$, $P(\delta'(\overline{\{1\}}))$,
$P(\delta(\{1\})) = \delta(\{1\}) = \delta'(\{1\}) + \delta'(\overline{\{1\}})$, $P(\delta'(\{1\}) + \delta'(\{2\})$,
and 12=6+3+3 facets repr. by $p_{ij} \geq p_{ii}$, $Tr_{ij,k} \geq p_{kk}$, $p_{ii} \geq 0$.

Anti-o-multicut quasi-semi-metrics

Given proper partition $\{S_1, \dots, S_t\}$, $2 \le t \le n$, of $\{1, \dots, n\}$, **anti-o-multicut q-s-metric** (or **anti-o-multicut**) $\alpha'(S_1, \dots, S_t)$ is $1 - \delta'_{ij}(S_1, \dots, S_t)$ if $1 \le i \ne j \le n$ and $= 0$, else.

It is a $\{0, 1\}$-valued q-s-metric, which is weightable iff $t{=}2$ (i.e. for **anti-o-cut** $\alpha'(S, \overline{S})$) with weight function $w(x) = 1_{x \in S}$.

Anticut semi-metric

$\alpha(S_t, \dots, S_1){=}\alpha'(S_1, \dots, S_t){+}\alpha'(S_t, \dots, S_1)$ (twice symmetrization) is graph path-metric $d(K_{|S_1|, \dots, |S_t|})$.

Anti-o-multicut quasi-semi-metrics

Given proper partition $\{S_1, \dots, S_t\}$, $2 \le t \le n$, of $\{1, \dots, n\}$, **anti-o-multicut q-s-metric** (or **anti-o-multicut**) $\alpha^{'}(S_1, \dots, S_t)$ is $1 - \delta^{'}_{ij}(S_1, \dots, S_t)$ if $1 \le i \ne j \le n$ and $= 0$, else.

It is a $\{0, 1\}$-valued q-s-metric, which is weightable iff t=2 (i.e. for **anti-o-cut** $\alpha^{'}(S, \overline{S})$) with weight function $w(x) = 1_{x \in S}$.

Anticut semi-metric
$\alpha(S_t, \dots, S_1) = \alpha^{'}(S_1, \dots, S_t) + \alpha^{'}(S_t, \dots, S_1)$ (twice symmetrization) is graph path-metric $d(K_{|S_1|, \dots, |S_t|})$.

For **semi-metrics**, $SMET_n = CUT_n$ if $n \le 4$, and all extreme rays of $SMET_5$ are all $2^4 - 1$ non-zero cuts and all $\binom{5}{2}$ anticuts $\alpha(\{a_1, a_2\}, \{a_3, a_4, a_5\})$ (permutations of $d(K_{2,3})$).

Are $\alpha^{'}$, except $\alpha^{'}(\{1\}, [n] \backslash \{1\}) = \sum_{s=2}^{n} \delta^{'}(\{s\}, [n] \backslash \{s\})$ and $\alpha^{'}(\{1\}, \dots, \{n\}) = \delta^{'}(\{n\}, \dots, \{1\})$, extreme in $QSMET_n$?

Extreme rays of $QSMET_4$, $QSMET_5$

$QSMET_4$ has 164 extreme rays in 10 orbits. Among 8 **$\{0,1\}$-valued ones** (116 ext. rays of $BQSMET_4$), 5 are of $\neq 0$ **o-multicuts** (74 ext. rays of $OMCUT_4$), including o-cuts $\delta'(\{1\})$, $\delta'(\{1,2\})$ (14 ext. rays of $OCUT_4$), and 3 of **anti-o-multicuts** $\alpha'(\{1,2\},\{3,4\})$, $\alpha'(\{1\},\{2\},\{3,4\})$, $\alpha'(\{1\},\{2,3\},\{4\})$.

Extreme rays of $QSMET_4$, $QSMET_5$

$QSMET_4$ has 164 extreme rays in 10 orbits. Among 8 $\{0,1\}$**-valued ones** (116 ext. rays of $BQSMET_4$), 5 are of $\neq 0$ **o-multicuts** (74 ext. rays of $OMCUT_4$), including o-cuts $\delta'(\{1\})$, $\delta'(\{1,2\})$ (14 ext. rays of $OCUT_4$), and 3 of **anti-o-multicuts** $\alpha'(\{1,2\},\{3,4\})$, $\alpha'(\{1\},\{2\},\{3,4\})$, $\alpha'(\{1\},\{2,3\},\{4\})$.

$QSMET_5$ has 229 orbits of extreme rays. Among 29 $\{0,1\}$**-valued ones**, 9 are of all **o-multicuts** $\delta'(S_1,\dots,S_t)\neq 0$ (including $\delta'(\{1\})$, $\delta'(\{1,2\})$) and 7 are of **anti-o-multicuts**.
Only 3 $\{0,1\}$-valued ones consist of **weightable q-s-metrics**:
2 above orbits of o-cuts and one of anti-o-cuts $\alpha'(\{1,2\})$.

Cones $PSMET_n$ and l_1-$PSMET_n$

cone	dim.	Nr. of ext. rays (orbits)	Nr. of facets (orbits)	diam.
CUT_3=$SMET_3$	3	3(1)	3(1)	1; 1
CUT_4=$SMET_4$	6	7(2)	12(1)	1; 2
CUT_5	10	15(2)	40(2)	1; 2
$SMET_5$	10	25(3)	30(1)	2; 2
CUT_6	15	31(3)	210(4)	1; 3
$SMET_6$	15	296(7)	60(1)	2; 2
l_1-$PSMET_3$=$PSMET_3$	6	13(5)	12(3)	
l_1-$PSMET_4$	10	44(9)	46(5)	
$PSMET_4$	10	62(11)	28(3)	
l_1-$PSMET_5$	15	166(14)	585(15)	
$PSMET_5$	15	1696(44)	55(3)	
l_1-$PSMET_6$	21	705(23)		
$PSMET_6$	21	337092(734)	96(3)	

$\{0,1\}$-valued partial semi-metrics

All such elements of $PSMET_n$ are $\sum_{0\le i\le n}\binom{n}{i}B(n-i)$ elements ($\sum_{0\le i\le n} Q(i)$ orbits under $Sym(n)$) of the form $J(S_0)+\delta(S_0,S_1,\dots,S_t)=P(\sum_{1\le i\le t}\delta'(S_i))$, where S_0 is any subset of $[n]=\{1,\dots,n\}$ and $S_1,\dots,S_t$ is any partition of $\overline{S_0}$.
$2^{n-1}+\sum_{1\le i\le n-1}\binom{n}{i}B(n-i)$ among them ($1+\lfloor\frac{n}{2}\rfloor+\sum_{1\le i\le n-1} Q(i)$ orbits) represent extreme rays: ones with $t=2$ if $S_0=\emptyset$ (w.l.o.g. suppose $S_i\ne\emptyset$ for $1\le i\le t$).

Here **partition number** $Q(i)$ is the number of ways to write i as a sum of positive integers;
Bell number $B(i)$ is the number of partitions (multicuts) of $[i]$, while the numbers of cuts $=2^{i-1}$, of o-cuts $=2^i$, of o-multicuts is **ordered Bell number** $Bo(i)$ of ordered partitions of $[i]$.

$\{0,1\}$-valued partial semi-metrics

See below $p=((p_{ij}))=J(\{67\})+\delta(\{1\},\{23\},\{45\},\{67\})=P(q)$ ($0,1$-valued extreme ray of $PSMET_7$) and its quasi-semi-metric $q=((q_{ij}=p_{ij}\text{-}p_{ii}))=\delta(\{1\})+\delta(\{23\})+\delta(\{45\})+\delta(\{67\})$ ($\{0,1\}$-valued non-extreme ray of $WQSMET_7$).

0	1	1	1	1	1	1	**0**	1	1	1	1	1	1
1	**0**	**0**	1	1	1	1	1	**0**	0	1	1	1	1
1	**0**	**0**	1	1	1	1	1	0	**0**	1	1	1	1
1	1	1	**0**	**0**	1	1	1	1	1	**0**	0	1	1
1	1	1	**0**	**0**	1	1	1	1	1	0	**0**	1	1
1	1	1	1	1	**1**	**1**	0	0	0	0	0	**0**	0
1	1	1	1	1	**1**	**1**	0	0	0	0	0	0	**0**

Unique orbit of **simplicial** (belong to $\binom{n+1}{2}$-1 facets) $\{0,1\}$-valued extreme rays of $PSMET_n$ consists of n rays $\sum_{1,i\neq j}^{n}\delta'(\{i\})$, $1\leq j\leq n$, i.e. $J(\{j\})+\delta(\{j\},S_1,\dots,S_{n-1})$ with all $|S_i|=1$.

Facets of l_1-$PSMET_n$

Let $b = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ and $\sum(b) = \sum_{i=1}^n b_i \in \{0, 1\}$. Then **hypermetric inequality** $Hyp_p(b): \sum_{1 \le i,j \le n} b_i b_j p_{ij} \le \sum_{i=1}^n b_i p_{ii}$ and, for $\max_{1 \le i \le n} |b_i| \le 2$, **modular inequality**

$$A_p(b): \sum_{1 \le i,j \le n} b_i b_j p_{ij} \le \sum_{i=1, b_i \ne 0}^n (2 - |b_i|) p_{ii}$$

are valid, for any $p = ((p_{ij})) \in l_1$-$PSMET_n$.

$PSMET_n$ has 3 orbits of facets, represented by $p_{ii} \ge 0$, $Hyp_p(1, -1, 0, \ldots, 0)$ and $Hyp_p(1, 1, -1, 0, \ldots, 0)$.

Facets of l_1-$PSMET_n$

Let $b = (b_1, \dots, b_n) \in \mathbb{Z}^n$ and $\sum(b) = \sum_{i=1}^n b_i \in \{0, 1\}$. Then **hypermetric inequality** $Hyp_p(b) : \sum_{1 \le i,j \le n} b_i b_j p_{ij} \le \sum_{i=1}^n b_i p_{ii}$ and, for $\max_{1 \le i \le n} |b_i| \le 2$, **modular inequality**

$$A_p(b) : \sum_{1 \le i,j \le n} b_i b_j p_{ij} \le \sum_{i=1, b_i \ne 0}^n (2 - |b_i|) p_{ii}$$

are valid, for any $p = ((p_{ij})) \in l_1$-$PSMET_n$.

$PSMET_n$ has 3 orbits of facets, represented by $p_{ii} \ge 0$, $Hyp_p(1, -1, 0, \dots, 0)$ and $Hyp_p(1, 1, -1, 0, \dots, 0)$.
l_1-$PSMET_3$=$PSMET_3$.
l_1-$PSMET_4$, besides 3 orbits of $PSMET_4$ has 2 orbits of facets, represented by $Hyp_p(1, 1, -1, -1)$, $A_p(2, 1, -1, -1)$.
l_1-$PSMET_5$, besides 3 orbits of $PSMET_5$, has 12 orbits of facets including represented by $Hyp_p(b)$ with $b = (1, 1, 1,-1,-1)$, $(1, 1,-1,-1, 0)$, $(1, 1, 1,-1,-2)$, $(2, 1,-1,-1,-1)$ and $A_p(b)$ with $b = (2, 1,-1,-1, 0)$, $(2, 2,-1,-1,-1)$, $(2, 1, 1,-1,-2)$, $(3, 1,-1,-1,-1)$.

Generalities on oriented n-cubes

We consider only **oriented** (or **unidirectional**) **n-cubes**, since there is no bidirectional electrical/optical converter and full-duplex transmission in optical fiber networks is costly.
The number of all orientations of n-cube $H(n)$ is $2^{n2^{n-1}}$.

Robbins, 1939: connected graph has **strong orientation** (i.e. strongly connected) if and only if it is bridgeless.
The number of strong orientations of n-cube is unknown.

Generalities on oriented n-cubes

We consider only **oriented** (or **unidirectional**) ***n*-cubes**, since there is no bidirectional electrical/optical converter and full-duplex transmission in optical fiber networks is costly.
The number of all orientations of n-cube $H(n)$ is $2^{n2^{n-1}}$.

Robbins, 1939: connected graph has **strong orientation** (i.e. strongly connected) if and only if it is bridgeless.
The number of strong orientations of n-cube is unknown.

In n-cube (as in any oriented bipartite graph), any 2 directed paths joining two fixed points have lengths equal modulo 2.
So, **symmetrization** $\frac{q(x,y)+q(y,x)}{2}$ of quasi-metric $q{=}q(Q(n))$ of any its strong orientation $Q(n)$ is integer-valued.

A vertex i in a n-cube is called **even** if its binary expansion has even number of ones and **odd**, otherwise.

O-diameter of oriented n-cube

Given a graph of diameter d and its strong orientation O, **oriented diameter** (or **o-diameter**) D_O is maximal length of shortest directed (u, v)-path.

Clearly, $D_O \geq d$; orientation O called **tight** if $D_O = d$.

Chvatal-Thomassen, 1978: $2d^2 + 2d \leq \max_O D_O \leq 5d^2 + d$.

Among strong orientations O of n-cube, $\min_O D_O = \infty, 3, 5$ and n for $n = 1, 2, 3$ and (McCanna, 1988) $n \geq 4$, respectively.

O-diameter of oriented n-cube

Given a graph of diameter d and its strong orientation O, **oriented diameter** (or **o-diameter**) D_O is maximal length of shortest directed (u,v)-path.

Clearly, $D_O \geq d$; orientation O called **tight** if $D_O = d$.

Chvatal-Thomassen, 1978: $2d^2 + 2d \leq \max_O D_O \leq 5d^2 + d$.

Among strong orientations O of n-cube, $\min_O D_O = \infty, 3, 5$ and n for $n = 1, 2, 3$ and (McCanna, 1988) $n \geq 4$, respectively.

For strong orientation O, $d(u,v){=}n$ implies $q_O(u,v){=}n$. It suffice to show $q_O(0, 2^n - 1){\leq}n$. For $1{\leq}i{<}n$, exists $\geq$1 arc (u,v) with $i, i{+}1$ ones in label $\{0,1\}$-expansions of u, v.

Everett-Gupta, 1989: there exists an acyclic (not strong) orientation of n-cube with finite length of shortest directed (u,v)-path $\geq F_{n+1}$ (Fibonacci number), i.e. $> (\frac{3}{2})^{n-1}$.

Connectivity

Given a digraph $D = (V, A)$, its **vertex-connectivity** κ (resp. **arc-connectivity** λ) is the minimum number of vertices (resp. arcs) needed to disconnect it. By Menger's theorem (max-flow-min-cut), κ (resp. λ) is minimum over $u, v \in V$ of the number of vertex- (resp. arc-) disjoint (u, v)-paths.

High connectivity of network D improve its fault-tolerance and communication performance (routing, broadcasting).

Connectivity

Given a digraph $D = (V, A)$, its **vertex-connectivity** κ (resp. **arc-connectivity** λ) is the minimum number of vertices (resp. arcs) needed to disconnect it. By Menger's theorem (max-flow-min-cut), κ (resp. λ) is minimum over $u, v \in V$ of the number of vertex- (resp. arc-) disjoint (u, v)-paths.

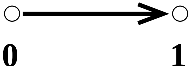
High connectivity of network D improve its fault-tolerance and communication performance (routing, broadcasting).

An **Hamilton (u, v)-path** in a graph is (u, v)-path visiting any vertex exactly once. In n-cube, it exists iff $d(u, v)$ is odd.
A graph is **k-vertex** (resp. **k-edge Hamiltonian**) if it remains Hamiltonian after deleting any k vertices (resp. edges).

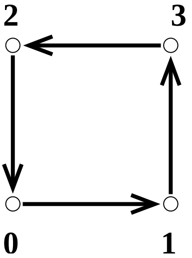
A (di)graph is **Eulerian** if exists a (directed) circuit visiting any (arc) edge exactly once; eqv., it is (strongly) connected and any vertex v has (indegree(v)=outdegree(v)) even degree.

Mini-cubes $Q(n)$

1-cube $Q(1)$ has two orientations.

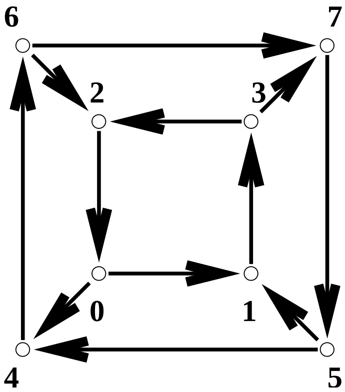


2-cube $Q(2)$ has two strongly connected orientations.



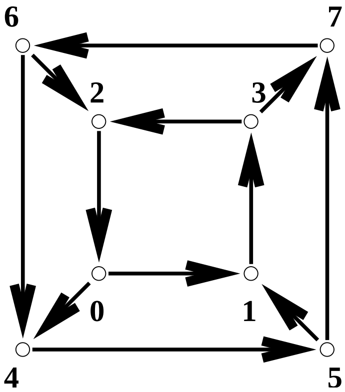
The **symmetrization** $D(Q(2)) = ((D_{ij}))=((\frac{1}{2}(q_{ij} + q_{ji})))$ of its quasi-metric $q = ((q_{ij}))$ is $2d(K_4)$, while $H(2) = C_4$.

3-cube: Chou-Du orientation $Q_{CD}(3)$



Chou-Du orientation $Q_{CD}(n)$ come from 2 factors $Q_{CD}(n-1)$ with mutually reversed orientations (above inside, outside squares $Q_{CD}(2)$) and, on remaining matching, arcs from each even vertex to its odd match. The symmetrization of its quasi-metric $q_{CD}(3)$ is $2d(K_8 - C_{0527} - C_{6341})$.

3-cube: Chou-Du orientation $Q_{CD'}(3)$



For odd $n \geq 3$, **2nd Chou-Du orientation** $Q_{CD'}(n)$ come from two factors $Q_{CD}(n-1)$ with the same orientation (above inside and outside squares $Q_{CD}(2)$) and, on remaining matching, again arcs from each even vertex to its odd match.

For even n, $Q_{CD'}(n) = Q_{CD}(n)$.

Chou-Du orientations CD, CD'

- Chou-Du, 1990: both $Q(n)$, as communication network (for high-speed computing using optical fibers as links), have efficient routing and short delay since are small:

  oriented diameter: $n+1$ for even n and $n+2$ for odd $n>1$ (for CD), 5 for $n=3$ and $n+1$ for other $n>1$ (for CD') and

  mean distance $\frac{n2^{n-1}+2n\binom{n-1}{\lfloor n/2 \rfloor}}{2^n-1}$, $\frac{n2^{n-1}+(n-1)\binom{n-1}{\lfloor n/2 \rfloor}+2}{2^n-1}$ (n odd).

Chou-Du orientations CD, CD'

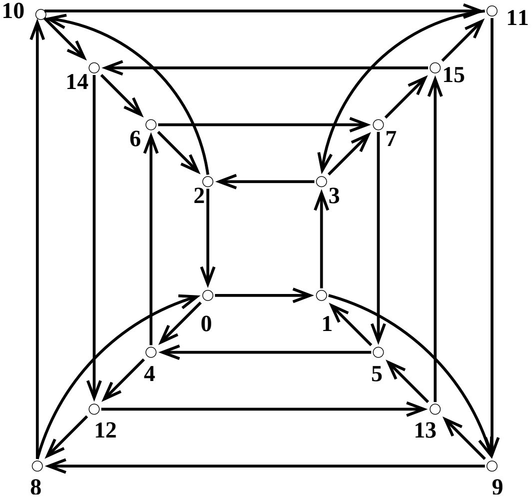
- Chou-Du, 1990: both $Q(n)$, as communication network (for high-speed computing using optical fibers as links), have efficient routing and short delay since are small:

  oriented diameter: $n+1$ for even n and $n+2$ for odd $n > 1$ (for CD), 5 for $n=3$ and $n+1$ for other $n > 1$ (for CD') and

  mean distance $\frac{n2^{n-1}+2n\binom{n-1}{\lfloor n/2 \rfloor}}{2^n-1}$, $\frac{n2^{n-1}+(n-1)\binom{n-1}{\lfloor n/2 \rfloor}+2}{2^n-1}$ (n odd).
- Let $C(x,y)$ be a largest set of vertex-disjoint (x,y)-paths (**max-container**), $L(C(x,y))$: longest path length in $C(x,y)$. **Wide-diameter**: $\max_{(x,y)} \min_{C(x,y)} L(C(x,y))$; $\geq$ o-diameter
- Jwo-Tuan, 1998: CD, CD' are maximally fault-tolerant, since $|C(x,y)| \leq \min(out(x), in(y))$ become equality.

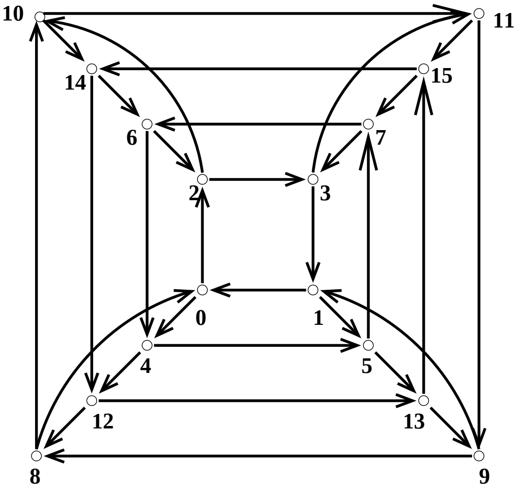
  Lu-Zhang, 2002: wide-diameters of CD, CD' are $n+2$.

Chou-Du orientation $Q_{CD}(4) = Q_{CD'}(4)$



4-cube: McCanna orientation $Q_{MC}(4)$

McCanna, 1988, gave this **tight** (i.e. with oriented diameter $n = 4$) orientation of 4-cube.



Generalized McCanna orientation

For $n \geq 4$, **generalized McCanna orientation** $Q_{MC}(n)$ come from 2 factors $Q_{MC}(n\text{-}1)$ with same orientation and, on remaining matching, arcs from each even vertex to its odd match.
A vertex i in a n-cube is called **even** if its binary expansion has even number of ones and **odd**, otherwise.

- Its **oriented diameter** is minimal: n, i.e. $Q_{MC}(n)$ is **tight**.
- Its **vertex**- and **arc-connectivity** are maximal: $\kappa = \lambda = \lfloor \frac{n}{2} \rfloor$.
- Fraigniaud-König-Lazard, 1992: it is **Hamiltonian** iff $n \geq 5$.

n-cube: signature-defined orientations

Given an orientation O of n-cube, its **signature** is ± 1-valued n-vector $a_O = (a_1, a_2, \dots, a_n)$ with $a_i = +1$ if the edge $(0, 2^i)$ is oriented in O by arc $(0, 2^i)$ and $a_i = -1$ if this edge is oriented by (incoming to 0) arc $(2^i, 0)$.
Excess of signature is the difference e between number of 1's and -1's in it. 0 is **source** if $e = n$ and **sink** if $e = -n$.

An orientation is **signature-defined** if any its arc is uniquely defined by arcs involving 0.

n-cube: signature-defined orientations

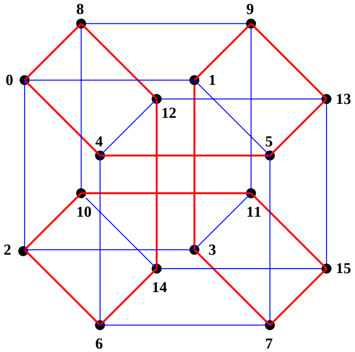
Given an orientation O of n-cube, its **signature** is ± 1-valued n-vector $a_O = (a_1, a_2, \dots, a_n)$ with $a_i = +1$ if the edge $(0, 2^i)$ is oriented in O by arc $(0, 2^i)$ and $a_i = -1$ if this edge is oriented by (incoming to 0) arc $(2^i, 0)$.
Excess of signature is the difference e between number of 1's and -1's in it. 0 is **source** if $e = n$ and **sink** if $e = -n$.

An orientation is **signature-defined** if any its arc is uniquely defined by arcs involving 0.
It is **$\|$-defined** if any its arc has the same orientation (from even to odd vertex) as the parallel edge involving 0.
Cariolaro: $\|$-defined orientation is strongly connected iff $|e| < n$.

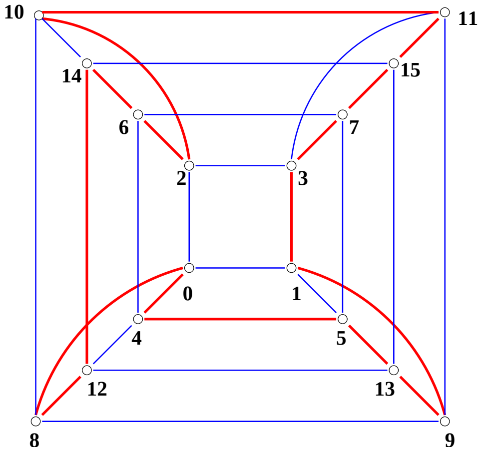
Chou-Du orientation CD is $\|$-defined, while CD', McCanna and Hamiltonian orientations are only signature-defined.

Hamiltonian decomposition of $H(n)$

Alspach-Bermond-Sotteau, 1990: edge-set of $H(n)$ can be decomposed into $\frac{n}{2}$ disjoint Hamilton cycles, if n is even, and into $\frac{n-1}{2}$ Hamilton cycles and a perfect matching, else.
For even n, $H(n) = C_4 \times \ldots \times C_4$ ($\frac{n}{2}$ times) $\sim$ 4-ary $\frac{n}{2}$-cube.
Stong, 2006: for odd n, **bidirected** Q_n decomposes into n **directed** Hamilton cycles.



Hamiltonian decomposition of $H(4)$



All Hamilton cycles of $H(4)$

Parkhomenko, 2001: 4-cube has 1344 Hamilton cycles.

See Hamilton cycle $V=\{v_i\}$, $1\leq i\leq 2^n$, as sequence $t(V)=\{1+\lg_2|t_i - t_{i+1}|\}$, $1\leq i\leq 2^n$, where t_i is label of v_i. Then (up to $Sym(4)$, reversals and cyclic shifts) all cycles are:

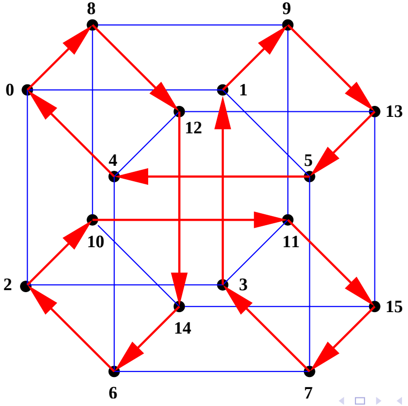
A $\{8,4,2,2\}$: 1 2 1 3 1 2 1 4 1 2 1 3 1 2 1 4;
B1 $\{6,6,2,2\}$: 1 2 1 3 2 1 2 4 1 2 1 3 2 1 2 4,
B2 $\{6,6,2,2\}$: 1 2 1 3 1 2 1 4 2 1 2 3 2 1 2 4;
C1 $\{6,4,4,2\}$: 1 2 1 3 2 1 2 4 3 1 3 2 1 3 1 4,
C2 $\{6,4,4,2\}$: 1 2 1 3 1 2 4 3 1 2 1 3 1 2 4 3,
C3 $\{6,4,4,2\}$: 1 2 1 3 2 1 2 4 1 3 1 2 3 1 3 4,
C4 $\{6,4,4,2\}$: 1 2 1 3 1 2 1 4 2 3 1 3 2 3 1 4,
C5 $\{6,4,4,2\}$: 1 2 1 3 1 2 4 2 1 3 1 2 1 3 4 3;
D $\{4,4,4,4\}$: 1 2 1 3 1 4 3 2 3 4 1 4 2 3 2 4.

Above **class** $\{a_1,\dots,a_n\}$ lists numbers a_i of i in a cycle.

The edges **not** belonging to Hamilton cycle form $C_8+C_4+C_4$, $C_6+C_6+C_4$, $C_{10}+C_6$ and $C_8+C_4+C_4$ for A, B2, C1 and C5.

Exp.: complementary Hamilton cycles

The sequence $t(V) = \{1 + \lg_2 |t_i - t_{i+1}|\}$, $1 \leq i \leq 2^4$, of red Hamilton cycle is given by: 4 3 2 4 3 4 1 3 4 3 2 4 3 4 1 3;
its permutation $(4, 3, 1, 2)$ is: 2 1 3 2 1 2 4 1 2 1 3 2 1 2 4 1,
a cyclic shift of which is B1: 1 2 1 3 2 1 2 4 1 2 1 3 2 1 2 4.
Remaining edges form ∼B1: 1 3 2 1 2 4 1 2 1 3 2 1 2 4 1 2.



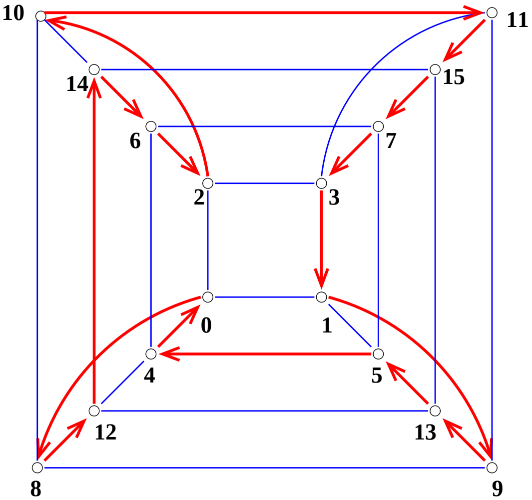
Hamilton orientations of n=2m-cube

For any $n = 2m$ and a decomposition of the edge-set of $2m$-cube into m disjoint Hamilton cycles, call **Hamilton orientation** any of 2^{m-1} orientations obtained by cyclically orienting those m cycles. Without loss of generality, orient 1st cycle arbitrary.

Any Hamilton orientation is signature-defined: number a_i uniquely identifies outcoming (if a_i=1) or incoming (if a_i=-1) to 0 Hamilton cycle and orientation on it. The number of 1's in its signature is $\frac{n}{2} = m$, i.e. its excess $e(a_O)$ is 0.

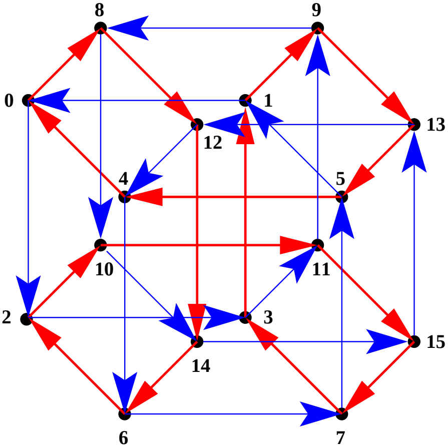
Orient arbitrarily 1st Hamilton cycle

Fix orientation of 1st (red) cycle and define orientation of 4-cube via orientation of 2nd (blue) Hamilton cycle.

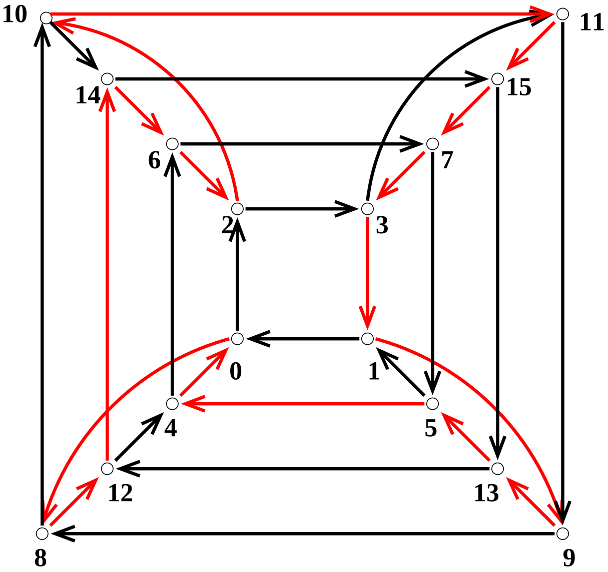


Hamilton orientation $Q_{B1}(4)$

The edge-set of $H(4)$ decomposed into two complementary Hamilton cycles with one (so, both) of type B1.
Orientation $Q_{B1}(4)$ is defined by signature $(-1, 1 - 1, 1)$.

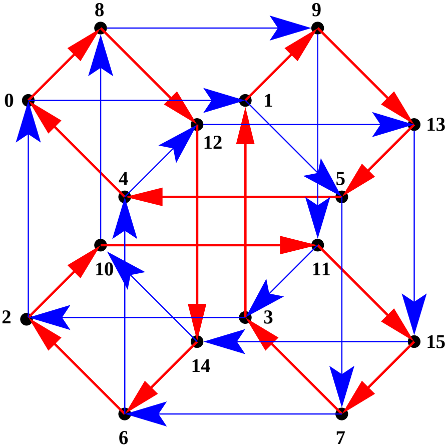


Hamilton orientation $Q_{B1}(4)$

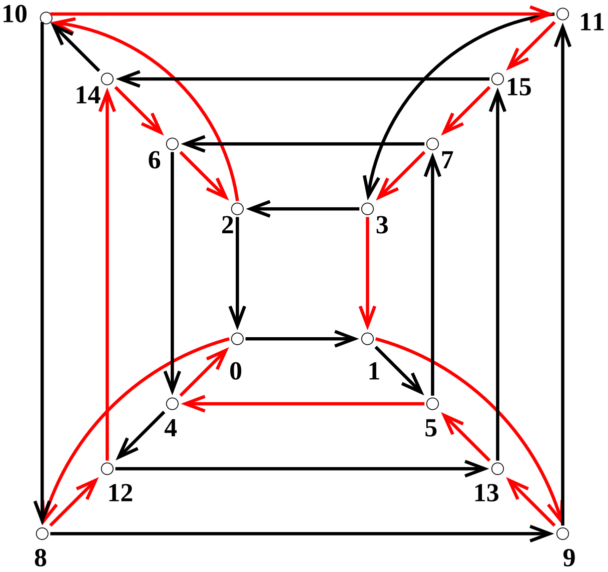


Hamilton orientation $Q_{B1'}(4)$

The edge-set of $H(4)$ decomposed into two complementary Hamilton cycles with one (so, both) of type B1.
Orientation $Q_{B1'}(4)$ is defined by signature $(1, -1 - 1, 1)$.



Hamilton orientation $Q_{B1'}(4)$



Ten Hamilton orientations of $H(4)$

Edge-complement of Hamilton cycle h of 4-cube is another Hamilton cycle h^* if and only if $h = B1, C2, C3, C4, D$; moreover, $h^* \sim h$ under $Sym(4)$, shifting and reversals.

Orient h so to get arc $(0,1)$ on it. Let O_h be orientation of $H(4)=$ $h+h^*$ with arc $(2,0)$ on h^* and by O'_h one with $(0,2)$.
So, signature is $(1,1,-1,-1)$ for all O_h, $(1,-1,-1,1)$ for O'_h with $h = B1, C1$ and $(1,-1,1,-1)$ for O'_h with $h = C3, C4, D$.

O-diameter is 6 for Q_{B1} and 5 for other 9. Q_{C3} has minimal, 4, $|\{(u,v) : q(u,v) = 5\}|$ and **mean** $q(u,v)$ (≈ 2.5); cf. 2 of $H(4)$.

Ten Hamilton orientations of $H(4)$

Edge-complement of Hamilton cycle h of 4-cube is another Hamilton cycle h^* if and only if $h = B1, C2, C3, C4, D$; moreover, $h^* \sim h$ under $Sym(4)$, shifting and reversals.

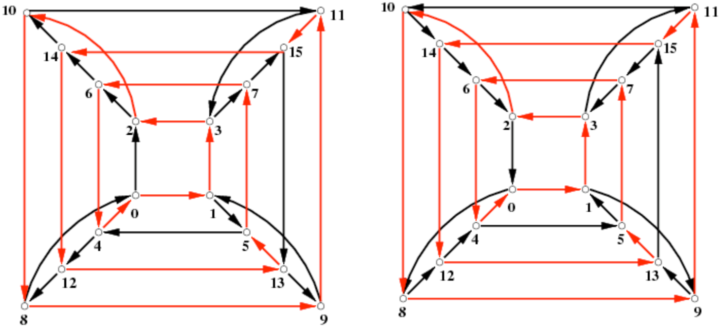
Orient h so to get arc $(0,1)$ on it. Let O_h be orientation of $H(4)= h+h^*$ with arc $(2,0)$ on h^* and by O'_h one with $(0,2)$.
So, signature is $(1,1,-1,-1)$ for all O_h, $(1,-1,-1,1)$ for O'_h with $h = B1, C1$ and $(1,-1,1,-1)$ for O'_h with $h = C3, C4, D$.

O-diameter is 6 for Q_{B1} and 5 for other 9. Q_{C3} has minimal, 4, $|\{(u,v) : q(u,v) = 5\}|$ and **mean** $q(u,v)$ (≈ 2.5); cf. 2 of $H(4)$.

Conjecture: for any m, there exists a Hamilton orientation of $H(2m)$ with $2^m d(K_4 \times K_4 \times \cdots \times K_4)$ (m times) being the symmetrization of its quasi-metric. It holds for 2-cube (unique strong orientation) and 4-cube (orientation Q_{B1}).
Remind that $H(2m) = C_4 \times C_4 \times \cdots \times C_4)$ (m times).

Hamilton orientations $O_B(4)$, $O_{B'}(4)$

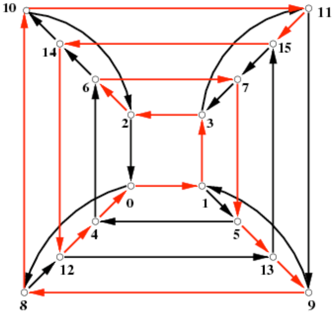
Each Hamilton cycle $V=\{v_i\}$, $1\leq i\leq 2^n$, as sequence $t(V)=\{1+\lg_2|t_i-t_{i+1}|\}$, $1\leq i\leq 2^n$, where t_i is label of v_i, is
B1 $\{6,6,2,2\}$: 1 2 1 3 2 1 2 4 1 2 1 3 2 1 2 4.

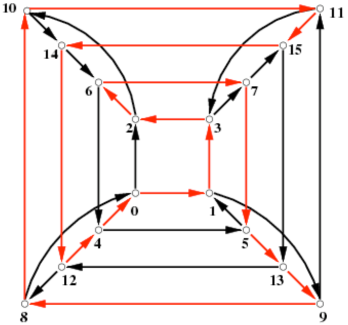


Hamilton orientations $O_{C2}(4)$, $O_{C2'}(4)$

Each cycle is **C2** $\{6, 4, 4, 2\}$: 1 2 1 3 1 2 4 3 1 2 1 3 1 2 4 3.
Wrapped grid G comes from $K_4 \times K_4$ on $((x_{ij}))$ by adding edges of $C_{11,22,33,44}$, $C_{12,21,43,34}$, $C_{13,24,42,31}$, $C_{14,23,41,32}$.

$2d(G)$ is symmetrization of quasi-metric of $O_{C2}(4)$.
This quasi-metric differs from one of Chou-Du $Q_{CD}(4)$ only by permutation $(4, 8)(5, 9)(6, 10)(7, 11)$ of vertices.



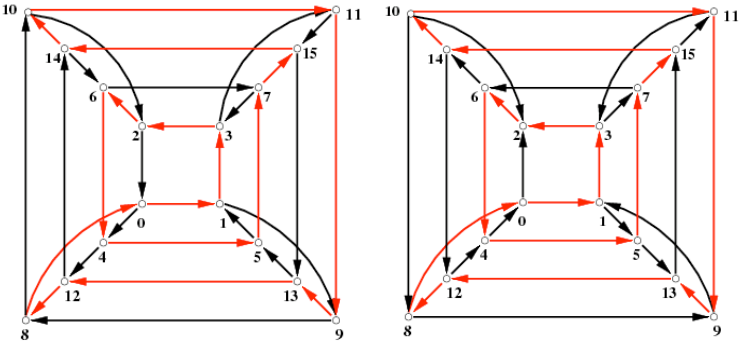


Hamilton orientations $O_{C3}(4)$, $O_{C3'}(4)$

Each Hamilton cycle $V=\{v_i\}$, $1\leq i\leq 2^n$, as sequence $t(V)=\{1+\lg_2|t_i-t_{i+1}|\}$, $1\leq i\leq 2^n$, where t_i is label of v_i, is

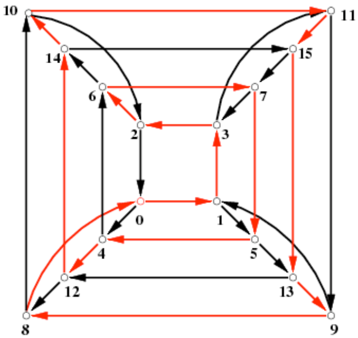
C3 $\{6,4,4,2\}$: 1 2 1 3 2 1 2 4 1 3 1 2 3 1 3 4.

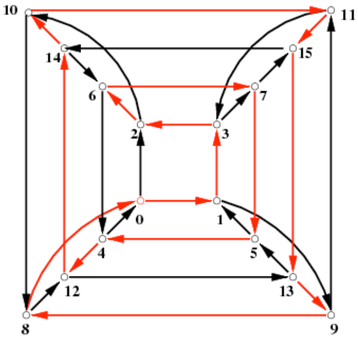
In $O_{C3}(4)$, $q(x,y)<5$ except $(x,y)=(2,10),(5,4),(11,3),(12,13)$.



Hamilton orientations $O_{C4}(4)$, $O_{C4'}(4)$

Each Hamilton cycle $V=\{v_i\}$, $1 \le i \le 2^n$, as sequence $t(V)= \{1+\lg_2|t_i - t_{i+1}|\}$, $1 \le i \le 2^n$, where t_i is label of v_i, is
C4 $\{6,4,4,2\}$: 1 2 1 3 1 2 1 4 2 3 1 3 2 3 1 4.

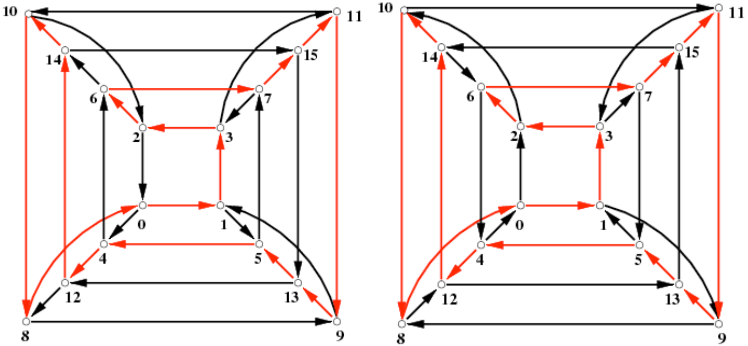




Hamilton orientations $O_D(4)$, $O_{D'}(4)$

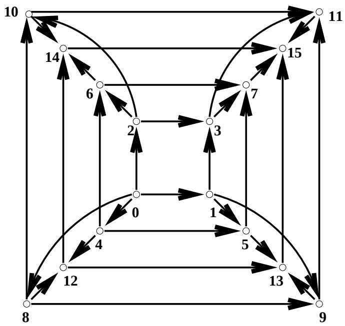
Each Hamilton cycle $V=\{v_i\}$, $1 \le i \le 2^n$, as sequence $t(V)$, is **D** $\{4,4,4,4\}$: 1 2 1 3 1 4 3 2 3 4 1 4 2 3 2 4.

In $O_D(4)$, $q(x,y)<5$ except $(x,y)=(0,14),(6,8),(10,4),(12,2)$ and $(3,13),(5,11),(9,7),(15,1)$. In $O_{D'}(4)$, $q(x,y)=5$ 10 times.



Inclusion (or Boolean) orientation $Q_I(n)$

Label vertices $0 \leq x \leq 2^n - 1$ of n-cube by subsets $A_x = \{1 \leq i \leq n : x_i = 1\}$ of $[n] = \{1, \dots, n\}$.
Inclusion orientation $Q_I(n)$: do arc AB if $A \subset B$ and $|B \backslash A|$=1.
Its **path quasi-semi-metric** is $|B \backslash A|$ if $A \subset B$ and $=\infty$, else,
while **measure q-s-metric** on $(\Omega = [n], \mathcal{A} = 2^{[n]}, \mu)$ is $\mu(B \backslash A)$.



Graph becomes strongly connected if add sink-souce arc $(2^n - 1, 0)$.

Unique-sink orientations

An orientation of n-cube is called **unique-sink orientation** if every face has unique sink.

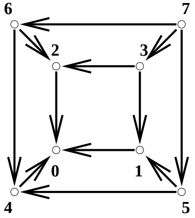
Examples:

1) the inclusion orientation $Q_I(n)$ and the arc-reversal of it on any fixed **matching** (set of disjoint edges) M of n-cube;

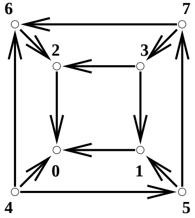
2) every acyclic orientation with unique-sink **on each 2-face**;

3) the **Klee-Minty orientation** $Q_{KM}(n)$: if the binary expansions of vertices $x, x' \in H(n)$ differ only in i-th position, then do arc (xx') if $\sum_{i \leq j \leq n} x_j$ is odd and arc $(x'x)$, otherwise.

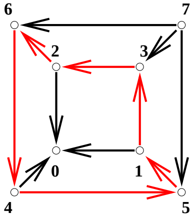
3-cube: some unique-sink orientations



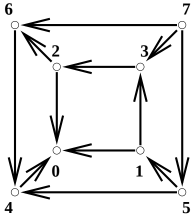
Inclusion orientation $Q_I(3)$



Klee-Minty orientation $Q_{KM}(3)$



(62,31,54)-reversed $Q_I(3)$



(62,31)-reversed $Q_I(3)$

Digression: Klee-Minty orientation

Klee-Minty orientation: if the binary expansions of vertices $x, x' \in H(n)$ differ only in i-th position, then do arc (xx') if $\sum_{i \le j \le n} x_j$ is odd and arc $(x'x)$, otherwise.

It is acyclic unique-sink orientation; moreover, each face has unique source.

It comes from combinatorial model (Avis-Chvatal, 1978) of **Klee-Minty cubes**, 1972, i.e., linear programs whose polytopes are deformed n-cubes (with skeleton of $H(n)$) but for which some pivot rules follow path through all 2^n vertices and hence, need exponential number of steps.

Some references on quasi-metrics

M.Charikar, K.Makarychev and Y.Makarychev, *Directed Metrics and Directed Graph Partitioning Problems*, Proc. of 17th ACM-SIAM Symposium on Discrete Algorithms (2006).

M.Deza and E.Deza, *Quasi-metrics, directed multicuts and related polyhedra*, European J. of Combinatorics, Special Issue " Discrete Metric Spaces", **21-6** (2000) 777–796.

M.Deza, M.Dutour and E.Deza, *Small cones of oriented semi-metrics*, American Journal of Mathematical and Management Sciences **22–3,4** (2003) 199–225.

P.Hitzler, *Generalized Metrics and Topology in Logic Programming Semantics*, PhD Thesis, Dept. Mathematics, National University of Ireland, Univ. College Cork, 2001. **S.G. Matthews**, *Partial metric topology*, Research Report 212. Dept. of Computer Science. Univ. of Warwick, 1992.

A.N.Patrinos and S.L.Hakimi, *Distance matrix of a graph and its tree realization*, Quart. Appl. Math. **30** (1972) 255–269.

References on oriented *n*-cubes

B. Alspach, J.C. Bermond and D.Sotteau, *Decompositions into cycles I: Hamilton decompositions*, in *Cycles and rays*, G.Hahn et al. (eds.) Kluwer Academic Press (1990) 9–18.

D. Avis and V. Chvatal, *Notes on Bland's Pivoting Rule*, Math. Programming Study, **8**(1978) 24–34.

G.Chartrand, D.Erwin, M.Raines, P.Zhang, *Orientation distance graphs*, Journal of Graph Theory **36-4** (2001) 230-241.

C-H.Chou and D.H.C.Du, *Uni-Directional Hypercubes*, in *Proc. Supercomputing '90*, (1990) 254–263.

H.Everett, A. Gupta, *Acyclic Directed Hypercubes may have Exponential Diameter*, Information Processing Letters **32-5** (1989) 243–245.

P.Fraigniaud, J-C.Knig, E.Lazard, *Oriented hypercubes*, Networks **39-2** (2002), 98–106.

J-S.Jwo and T-C.Tuan, *On container length and connectivity in unidirectional hypercubes*, Networks **32-4** (1998), 307-317.

References on oriented *n*-cubes

D.W.Krumme, *Fast Gossiping for the Hypercube*, SIAM J. Comput. **21-2** (1992) 365–380.

C.Lu and K.Zhang, *On container length and wide-diameter in unidirectional hypercubes*, Taiwanese Journal of Math. **6-1** (2002) 75–87.

J.Matousek, *The Number Of Unique-Sink Orientations of the Hypercube*, Combinatorica, **26-1** (2006) 91–99.

J.E.McCanna, em Orientations of the n -cube with minimum diameter, Discrete Mathematics **68** (1988) 309–310.

P.P.Parkhomenko, *Classification of the Hamiltonian Cycles in Binary Hypercubes*, Automation and Remote Control **62-6** (2001) 978–991.

H.E.Robbins, *A Theorem on Graphs with an Application to Traffic Control*, Amer. Math. Monthly **46** (1939) 281–283.

R.Stong, *Hamilton decompositions of directed cubes and products*, Discrete Mathematics **306-18** (2006) 2186–2204.