On Variants of Induced Matchings

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Distance-*k* Matchings

Let $G = (V,E)$ be a undirected finite simple graph. An edge set $M \subseteq E$ is a *matching in G* if the edges in M are mutually vertex-disjoint.

An edge set M is an *induced matching in G* [Kathie Cameron 1989] (also called *strong matching* [Golumbic, Laskar 1993])

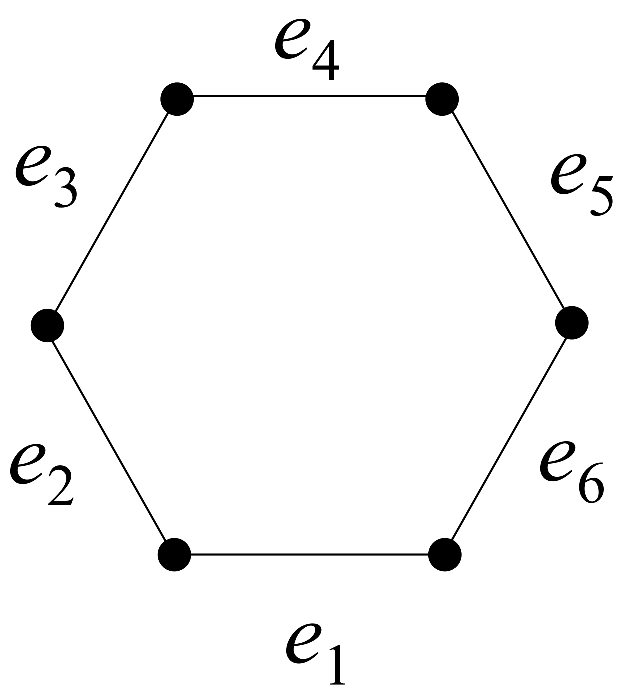
if the mutual distance of edges in M is ≥ 2.

Distance-*k* Matchings

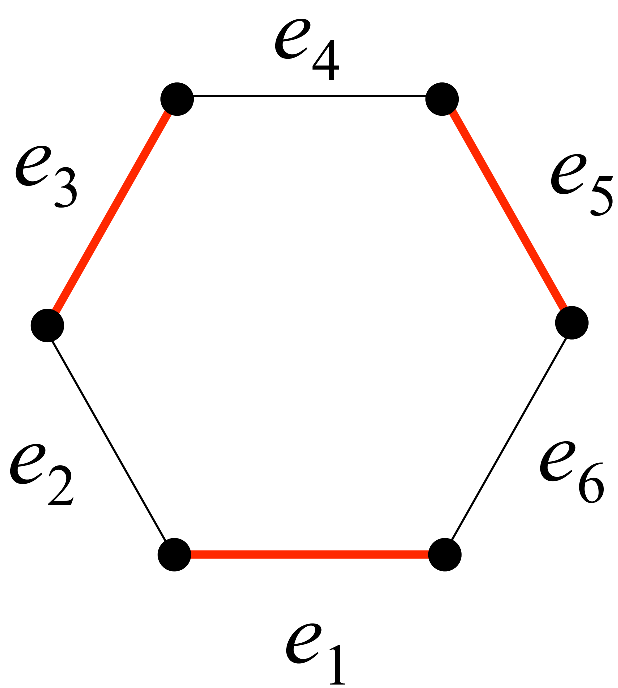
An edge set M is a *distance-k matching in G* if the mutual distance of edges in M is $\geq k$

(called *δ-separated matching* [Stockmeyer, Vazirani 1982]).

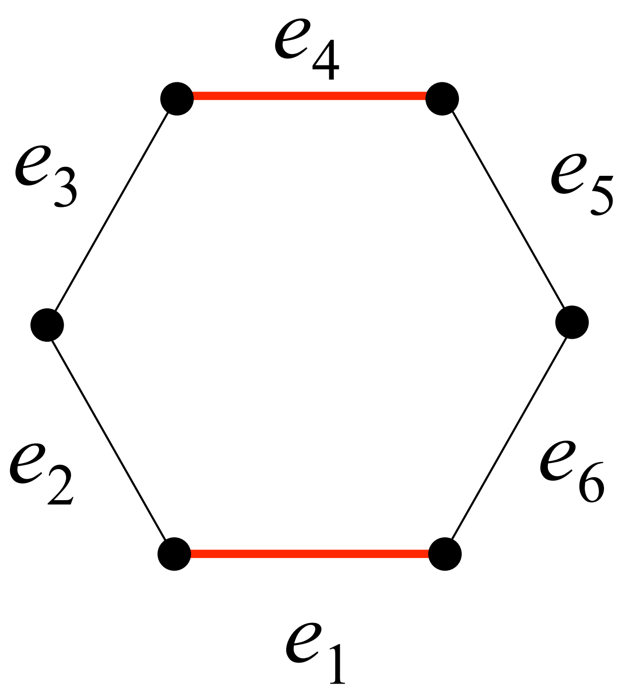
Matchings



Matchings



Induced Matchings



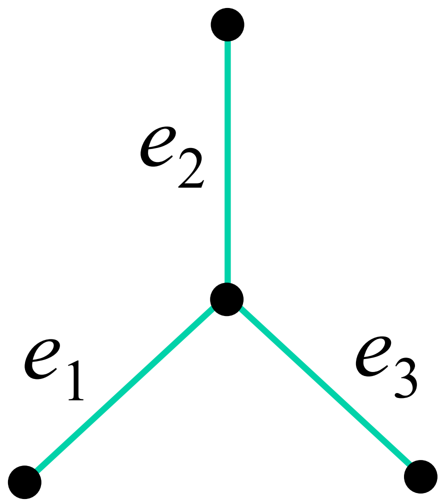
Line Graph

For graph $G = (V, E)$, let $L(G) = (E, E')$ with edges

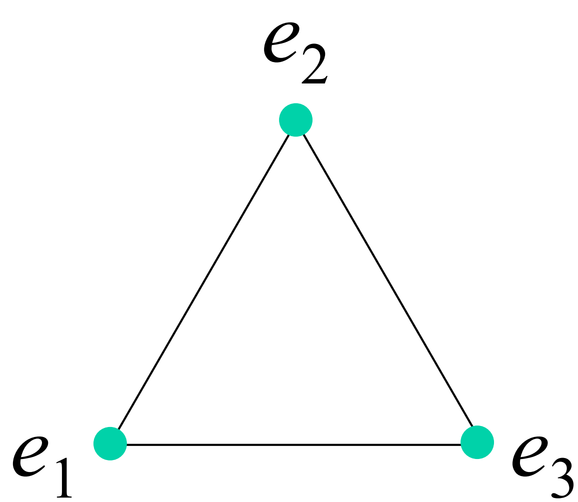
$$xy \in E' \Leftrightarrow x \cap y \neq \emptyset$$

denote the *line graph of G*.

Line Graph

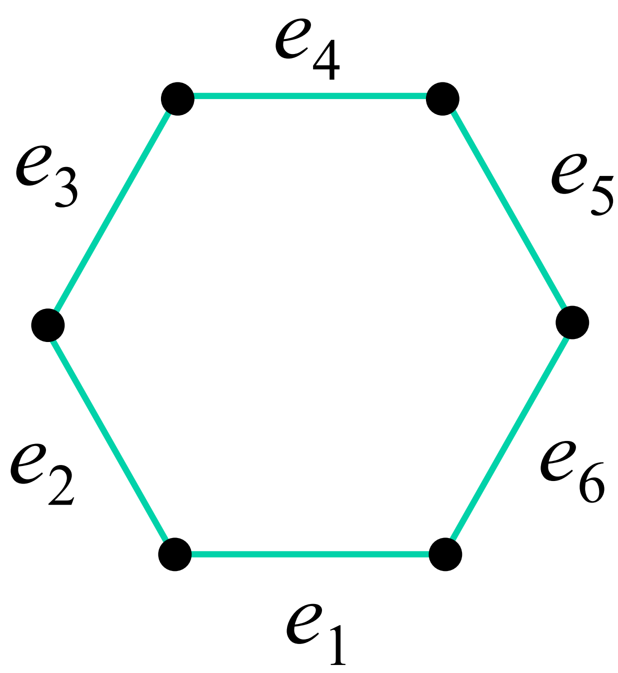


G



$L(G)$

Line Graph



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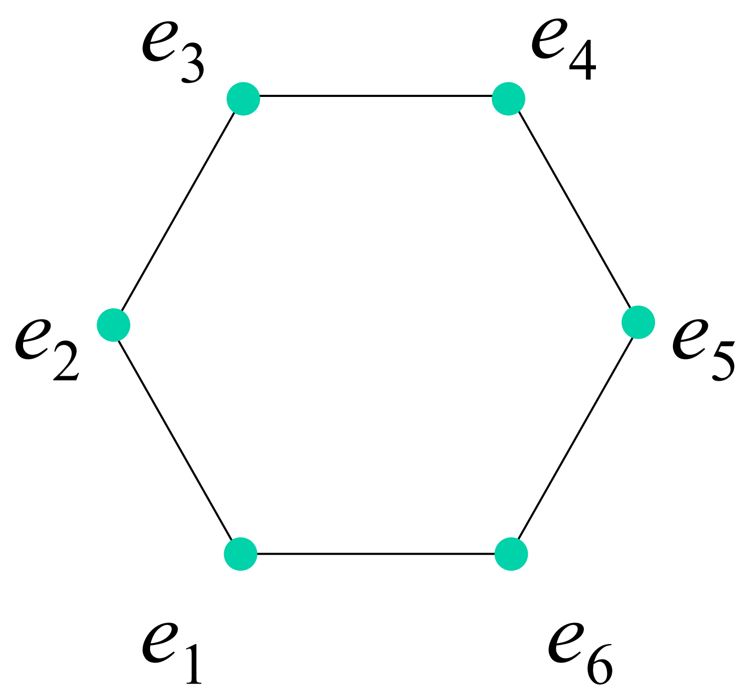
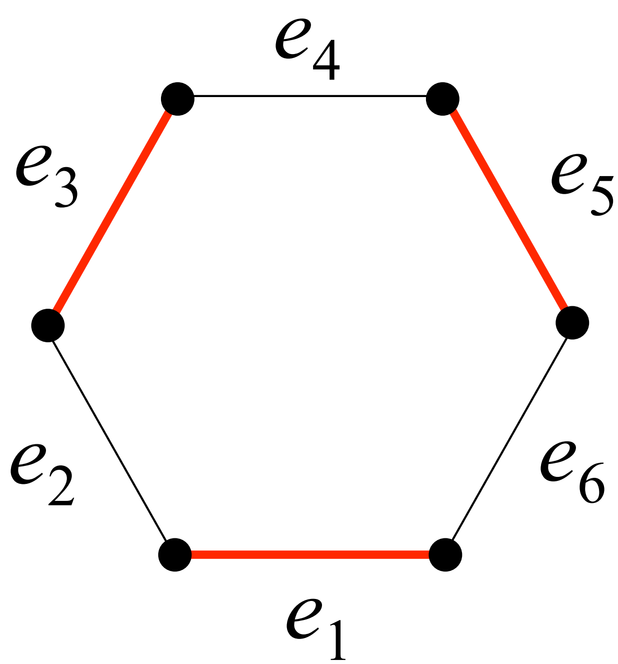
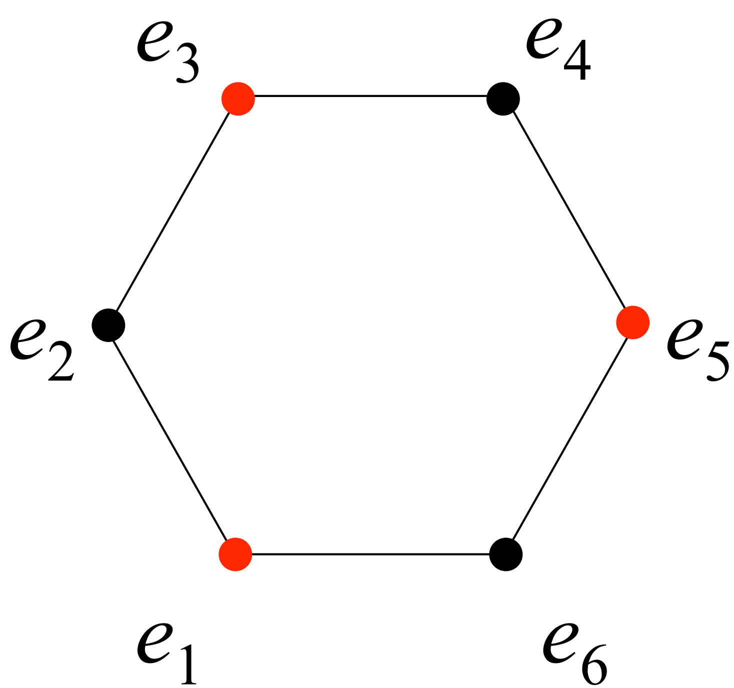


$L(G)$

Line Graph





Graph Powers

For graph $G = (V, E)$, let $G^k = (V, E^k)$ with

$$xy \in E^k \Leftrightarrow \mathrm{dist}_G(x,y) \leq k$$

denote the *k-th power of G*.

Induced Matchings

$L(G)^2$ is the *square of the line graph* of G, i.e., the vertex set of $L(G)^2$ is E, and two edges of G are adjacent in $L(G)^2$ if they share a vertex or are connected by an edge in G.

Induced Matchings

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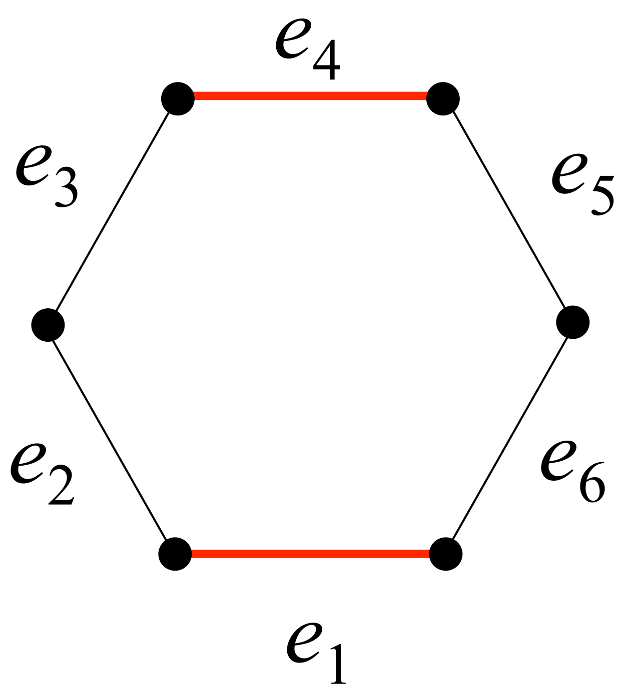
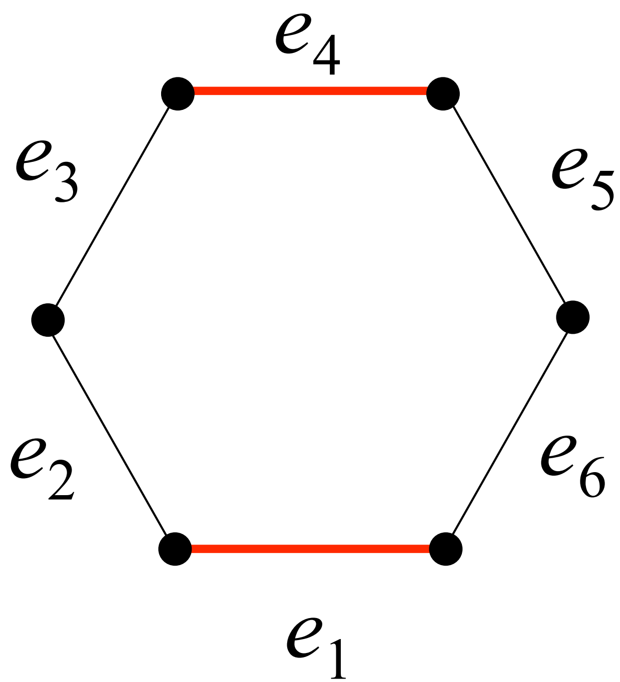
Fact.

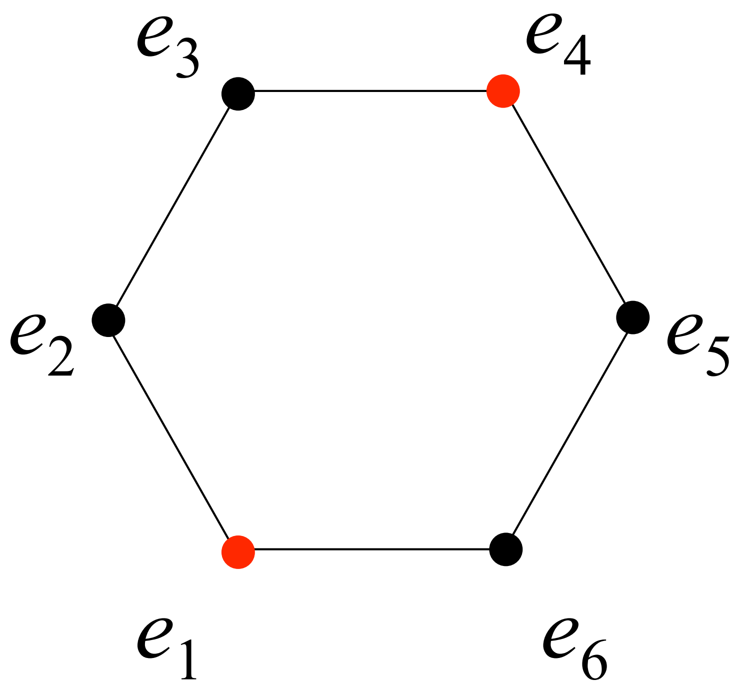
Induced matchings in G = independent vertex sets in $L(G)^2$.

Induced Matchings



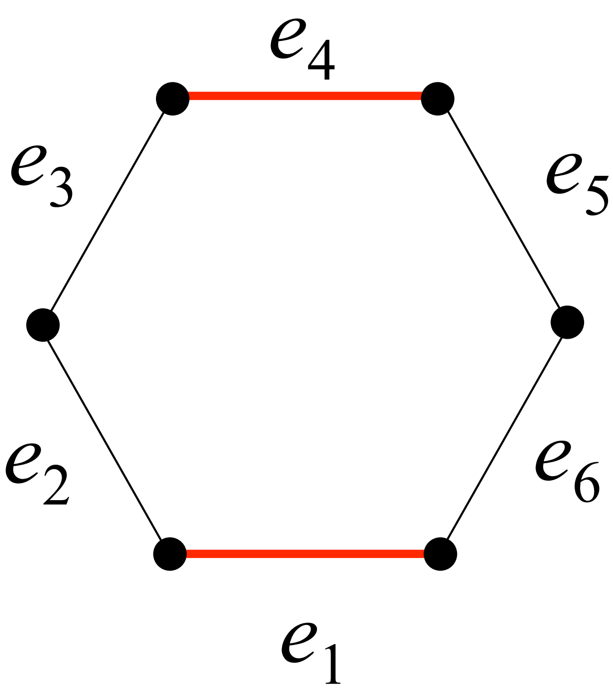
Induced Matchings

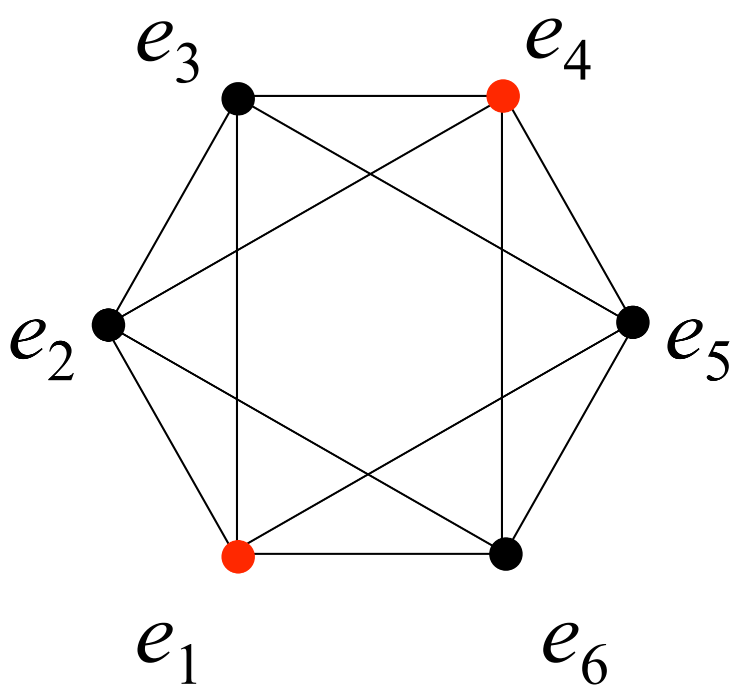




$L(G)$

Induced Matchings





$L(G)^2$

Maximum Induced Matchings

Maximum Matching Problem:

Find a maximum matching of largest size.

Maximum Induced Matching (MIM) Problem:

Find a max. induced matching of largest size.

NP-complete [Stockmeyer, Vazirani 1982, Kathie Cameron 1989]

Maximum Induced Matchings

The Maximum Induced Matching Problem remains NP-complete for

very restricted bipartite graphs [Ko, Shepherd 2003, Lozin 2002] and for

line graphs (and thus also for claw-free graphs) [Kobler, Rotics 2003].

Maximum Induced Matchings

- G chordal $\Rightarrow L(G)^2$ chordal [Cameron 1989].
- G circular-arc graph $\Rightarrow L(G)^2$ circular-arc graph [Golumbic, Laskar 1993]
- G cocomparability graph $\Rightarrow L(G)^2$ cocomparability graph [Golumbic, Lewenstein 2000]
- G weakly chordal $\Rightarrow L(G)^2$ weakly chordal [Cameron, Sritharan, Tang 2003]
- stronger result for AT-free graphs [J.-M. Chang 2004]

Maximum Induced Matchings

Hence: MIM in polynomial time for

- chordal graphs [Cameron 1989]
- circular-arc graphs [Golumbic, Laskar 1993]
- cocomparability and interval dimension *k* graphs [Golumbic, Lewenstein 2000]
- AT-free graphs [J.-M. Chang 2004]
- weakly chordal graphs [Cameron, Sritharan, Tang 2003]

Distance-k Matchings

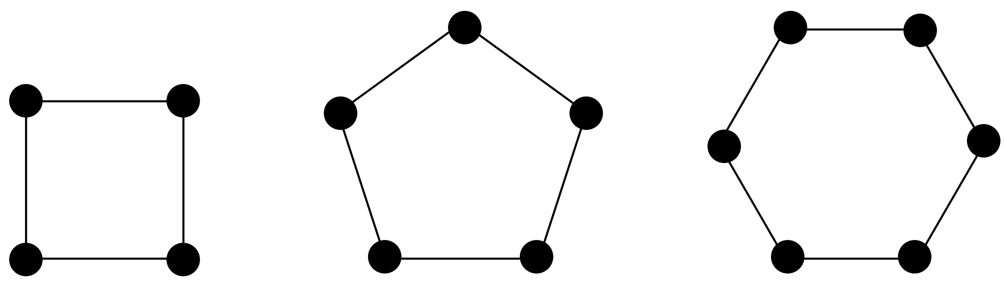
$L(G)^k$ is the *k-th power of the line graph* of G, i.e., the vertex set of $L(G)^k$ is E, and two edges of G are adjacent in $L(G)^k$ if their distance in $L(G)$ is at most k.

Fact.

Distance-k matchings in G = independent vertex sets in $L(G)^k$.

Chordal Graphs

Graph G is *chordal* if it contains no chordless cycles of length at least four.



Chordal Graphs

Graph G is *chordal* if it contains no chordless cycles of length at least four.

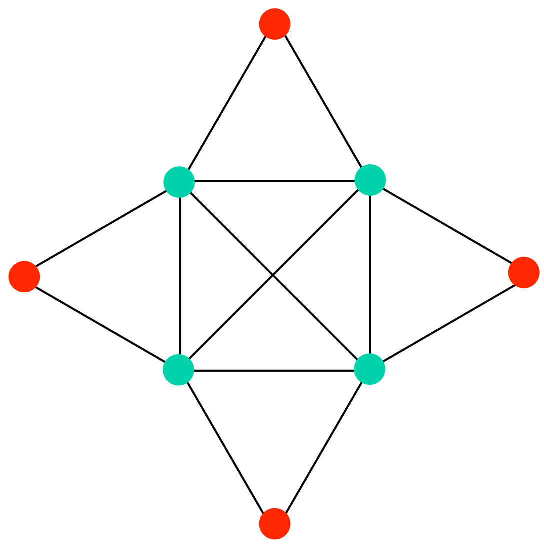
Chordal graphs have many facets:

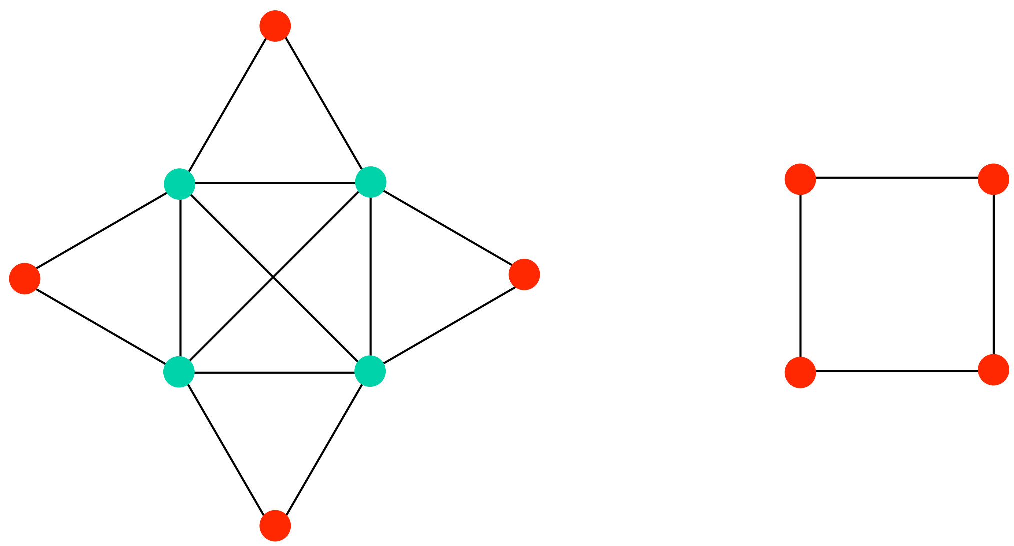
- clique separators
- clique tree
- simplicial elimination orderings
- intersection graphs of subtrees of a tree ...

Graph Powers

[Duchet, 1984]: Odd powers of chordal graphs are chordal.

But: Even powers of chordal graphs are in general not chordal.





Powers of Chordal Graphs

[K. Cameron, 1989]:

G chordal $\Rightarrow L(G)^2$ chordal.

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even in linear time! [B., Hoang 2005].

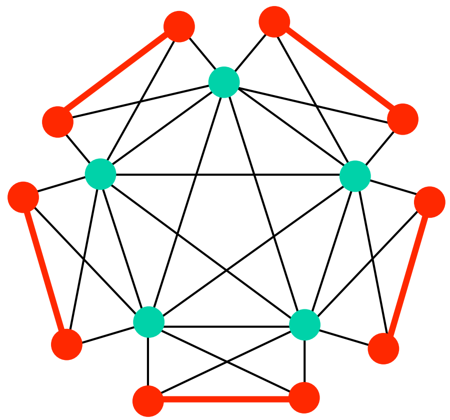
Powers of Chordal Graphs

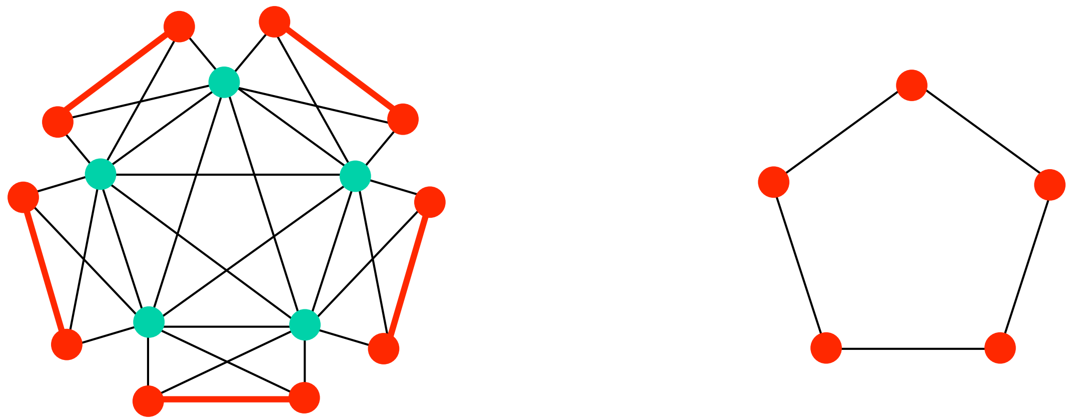
[K. Cameron, 1989]:

G chordal $\Rightarrow L(G)^2$ chordal

$\Rightarrow$ MIM problem in polynomial time on chordal graphs

But: If G is chordal then in general, $L(G)^3$ is not chordal.





Maximum Distance-3 Matchings

Theorem.

The Maximum Distance-3 Matching Problem for chordal graphs is NP-complete.

Maximum Distance-3 Matchings

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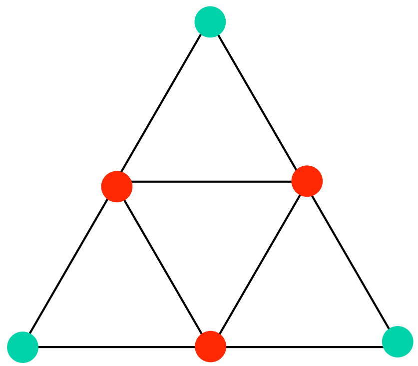
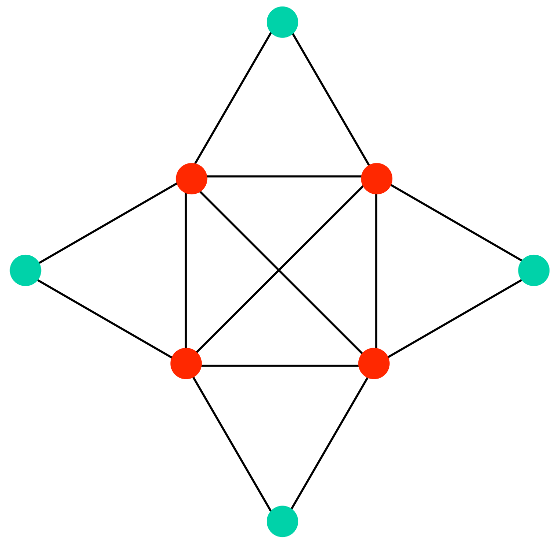
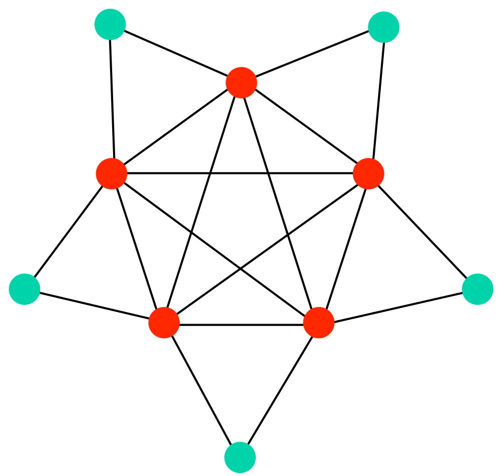
The Maximum Distance-3 Matching Problem for chordal graphs is NP-complete.

(Reduction from Maximum Independent Set for any graph [GT20])







Powers of strongly chordal graphs

A graph is *strongly chordal* if it is chordal and sun-free.

Theorem. [Lubiw 1982; Dahlhaus, Duchet 1987; Raychaudhuri 1992] For every $k \geq 2$:

G strongly chordal $\Rightarrow$ G^k strongly chordal.

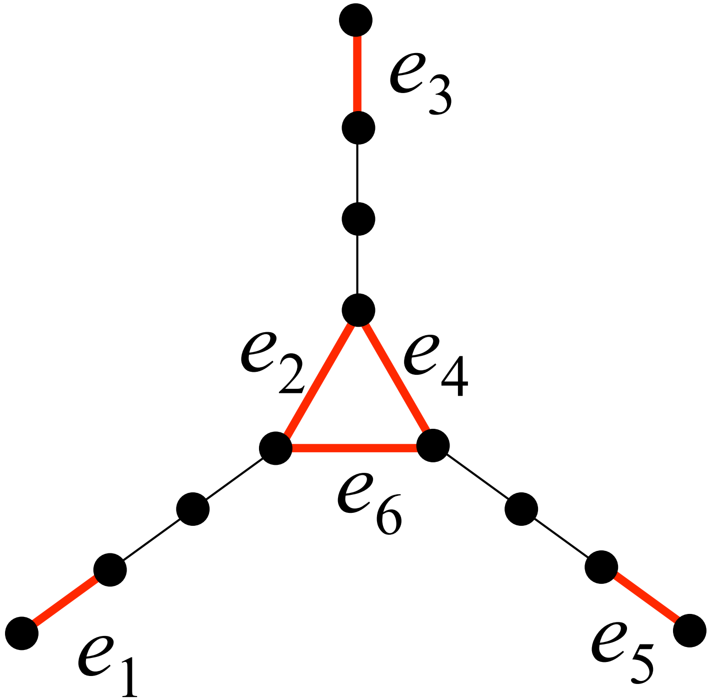
Powers of strongly chordal graphs

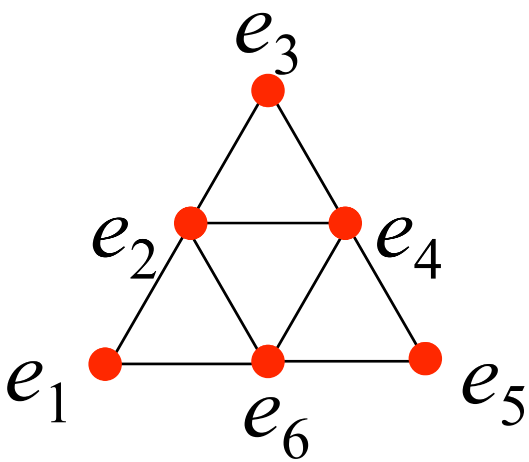
Recall: G chordal $\Rightarrow L(G)^2$ chordal

$\Rightarrow$ MIM problem in polynomial time on strongly chordal graphs.

But: If G is strongly chordal then in general, $L(G)^3$ is not strongly chordal.

$L(G)^3$ not strongly chordal





Maximum Distance-3 Matchings

Theorem.

If G is strongly chordal then $L(G)^3$ is chordal.

Maximum Distance-3 Matchings

Theorem.

If G is strongly chordal then $L(G)^3$ is chordal.

Corollary.

The Maximum Distance-k Matching Problem for strongly chordal graphs is solvable in polynomial time for every $k \geq 1$.

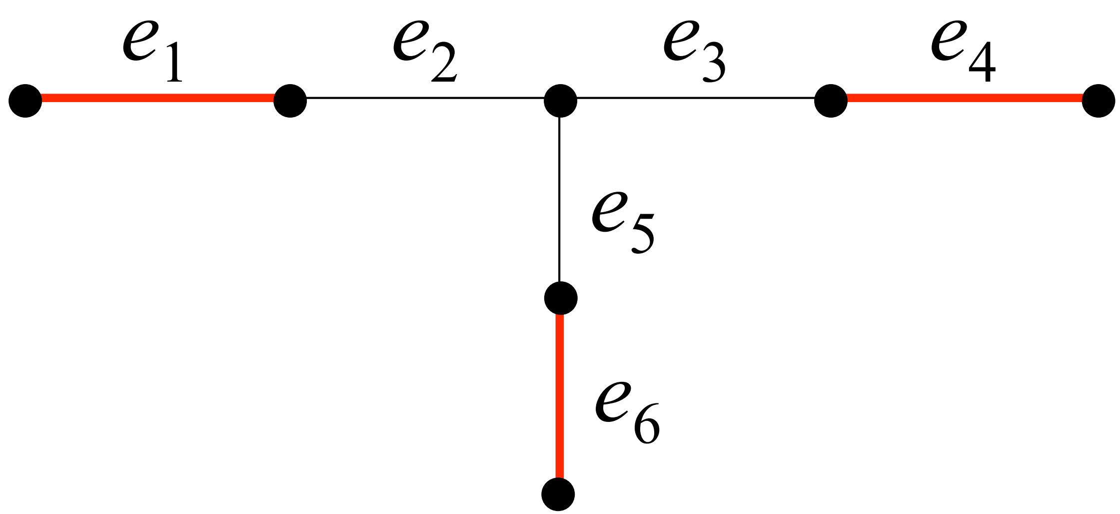
Dominating Induced Matchings

An induced matching M is a *dominating induced matching* (*d.i.m.*) in G if M intersects every edge in G. In other words:

- M is dominating in $L(G)$ and
- M is independent in $L(G)^2$.

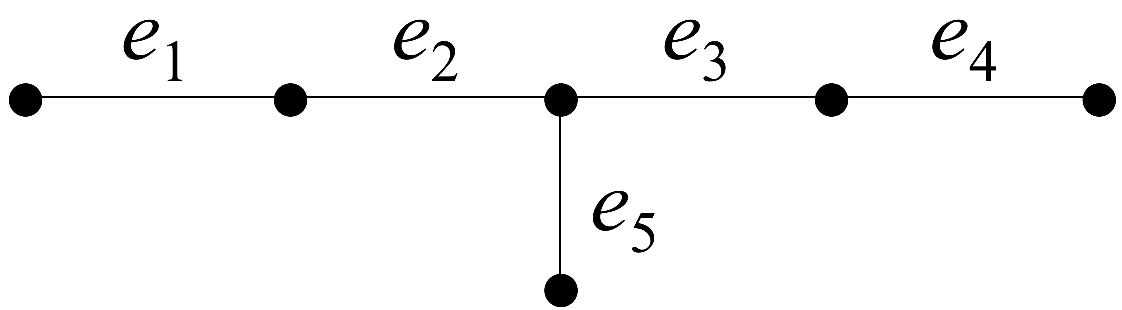
Dominating Induced Matchings

Example:



Dominating Induced Matchings

Note that there are graphs (even trees) without such an edge set:



Dominating Induced Matchings

The *Dominating Induced Matching* (*DIM*) Problem is:

Given a graph, does it have a dominating induced matching?

Also called the *Efficient Edge Domination* (*EED*) Problem.

Dominating Induced Matchings

Theorem [Grinstead, Slater, Sherwani, Holmes 1993]

The EED problem is NP-complete in general and efficiently solvable for series-parallel graphs.

Dominating Induced Matchings

Theorem [Lu, Tang 1998]

The EED problem is NP-complete for bipartite graphs, and is efficiently solvable for bipartite permutation graphs.

Theorem [Cardozo, Lozin 2008]

The EED problem is NP-complete for (very special) bipartite graphs, and is efficiently solvable for claw-free graphs.

Dominating Induced Matchings

Theorem [B., Nevries 2009]

The EED problem is solvable in

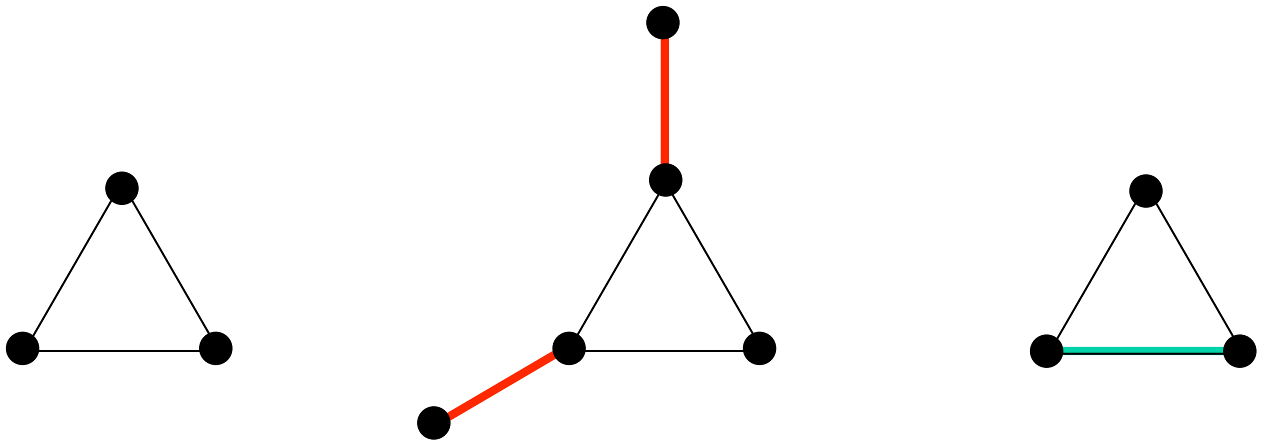
- linear time for chordal bipartite graphs;
- polynomial time for hole-free graphs.

Dominating Induced Matchings

Proposition.

Let M be a d.i.m. Then:

(i) M contains exactly one edge of every triangle.



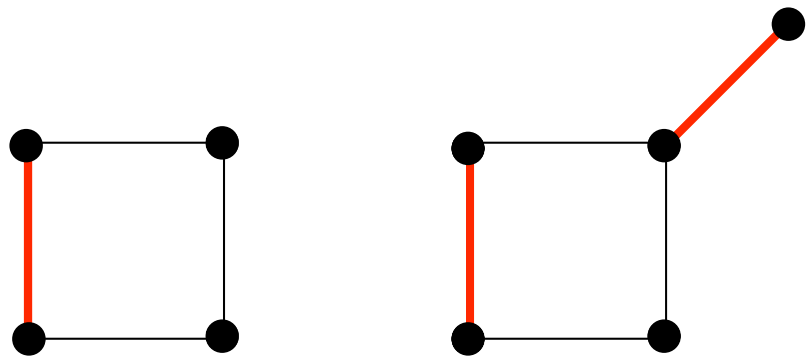
Dominating Induced Matchings

Proposition.

Let M be a d.i.m. Then:

(i) M contains exactly one edge of every triangle.

(ii) M contains no edge of any C_4.



Dominating Induced Matchings

Proposition.

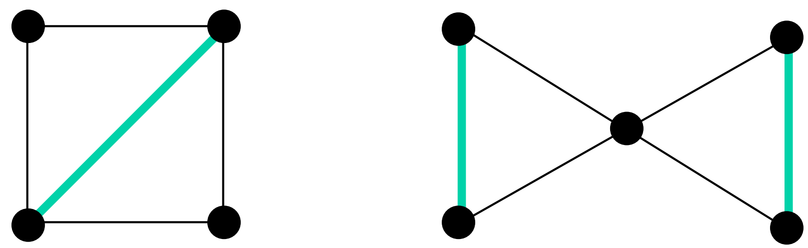
Let M be a d.i.m. Then:

(i) M contains exactly one edge of every triangle.

(ii) M contains no edge of any C_4.

(iii) Every mid-edge of a diamond is in M.

(iv) The peripheral edges of any butterfly are in M.



Dominating Induced Matchings

Proposition.

Let M be a dim. Then:

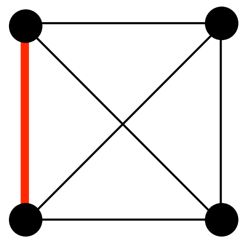
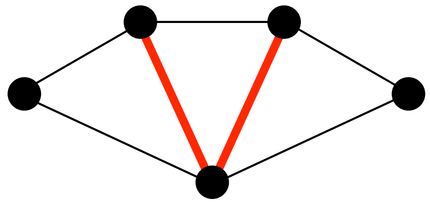
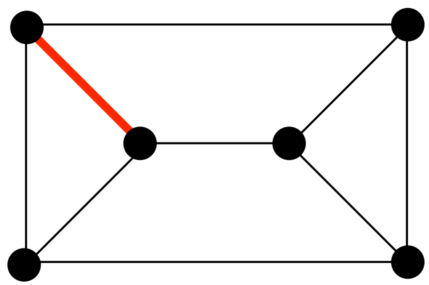
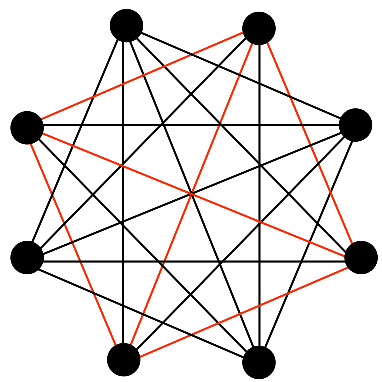
(i) M contains exactly one edge of every triangle.

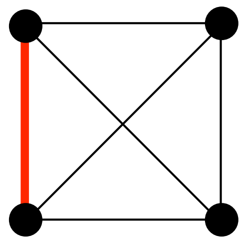
(ii) M contains no edge of any C_4.

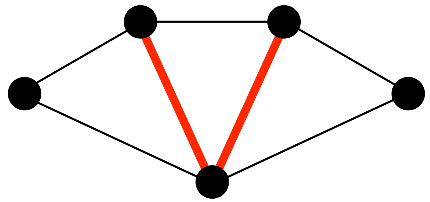
(iii) Every mid-edge of a diamond is in M.

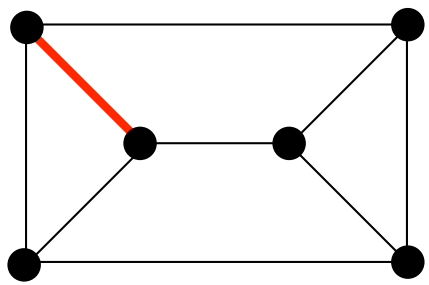
(iv) The peripheral edges of any butterfly are in M.

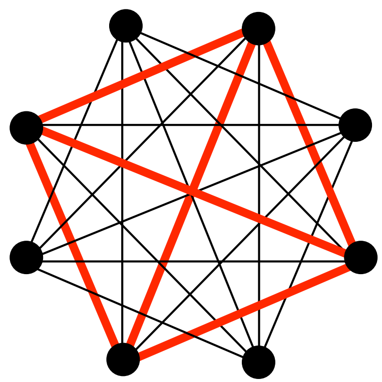
(v) Graphs with dim are (K_4,gem,long-antihole)-free.







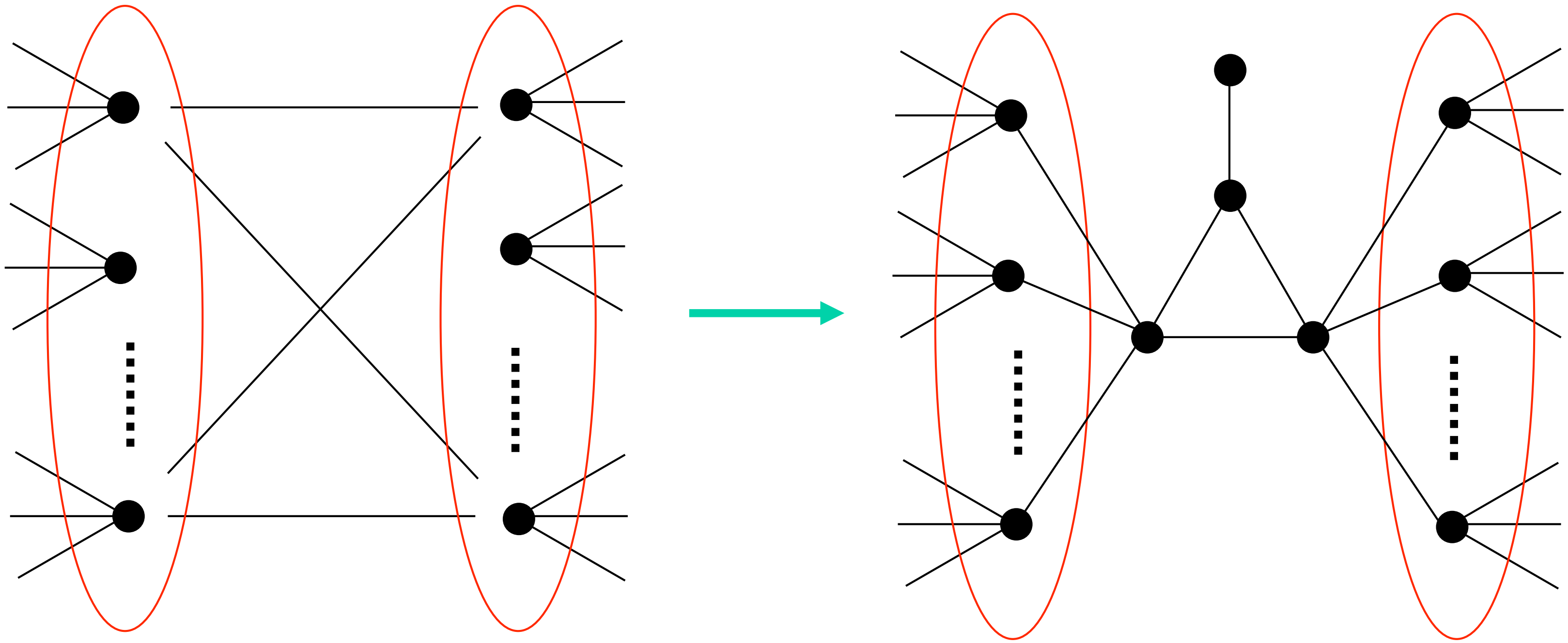


Dominating Induced Matchings

Lemma.

Let M be a d.i.m. in a chordal bipartite graph G, and let $Q = X \cup Y$ be a 2-connected component in G. Then either M dominates all vertices in X and none in Y or vice versa.

$\Rightarrow$ reduction to the EED problem in a (K_4-free) block graph G' via the following gadget:



Dominating Induced Matchings

Lemma.

A chordal bipartite graph G has a d.i.m. $M \Leftrightarrow$ the (K_4-free) block graph G' has a d.i.m. M', and the weights of M and M' coincide.

Dominating Induced Matchings

Note that hole-free graphs with d.i.m. are weakly chordal (no long antiholes).

Theorem.

For hole-free graphs, the minimum weight dominating induced matching problem can be solved in polynomial time.

Thank you for your attention!

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