Bounds on broadcast time in well-connected graphs

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Abstract

Given a graph G, the *broadcast time* $b(G)$ of G is the least number of time units required to disseminate one piece of information starting at an arbitrary originating vertex in G, where each transmission takes one time unit and each informed vertex can transmit the message to one of its neighbors at any time. We initiate the study of finding upper bounds on broadcast time $b(G)$ in highly connected graphs. In particular, we give upper bounds on $b(G)$ for k-connected graphs and graphs with a large minimum degree. We prove that $b(G) \leq \lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$ for all k-connected graphs. For many families of graphs this bound is tight. We also show that if the minimum degree of G is at least $n/2$, then $b(G) \leq \lceil \log n \rceil + 3$ time, and derive various similar upper bounds on graphs with large minimum degree. Finally, we discuss an open problem that relates the broadcast time of a graph with its minimum degree, when the latter is small.

Keywords: Broadcasting, k-connected graphs, minimum degree, diameter

1. Introduction

Broadcasting is a problem in which a sender, usually called the *originator*, has a piece of information and wishes to inform all network members.

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Let $G = (V, E)$ be an undirected loop-free connected graph representing a network. The process takes place with the following conditions [21]:

1. Time units are discrete
2. A *call* takes place when an informed vertex (a sender) informs one of its uninformed neighbors (a receiver)
3. Each call requires one time unit,
4. Each vertex can participate in only one call in each time unit,
5. In one time unit, multiple calls can be performed in parallel.
6. The process ends when all vertices are informed.

The set of calls used to distribute the message from originator v to all other vertices is called a *broadcast scheme* for vertex v. The broadcast scheme induces a spanning tree called a *broadcast tree* rooted at v and all the communication lines are labeled with the transmission time.

The *broadcast time* of a vertex $v \in V$ (denoted by $b(v, G)$) is the minimum time required to complete broadcasting when the originator is v. The *broadcast time* of the graph G (denoted by $b(G)$) is the maximum among all vertices' broadcast times i.e., $b(G) = \max_{v \in V(G)}\{b(v, G)\}$. If G is not connected, then obviously $b(G)$ is not finite. Since at each time unit at least one vertex must be informed, $n - 1$ is an upper bound for any connected graph. At each time unit the number of informed vertices can at most double which yields the lower bound of $\lceil \log n \rceil$ (all logarithms are implicitly assumed to be of base two in the paper). A graph that has $b(G) = \lceil \log n \rceil$ is called a *broadcast graph*. Given a diametral path in the graph, there is no shorter path to inform one endpoint if the originator is the other endpoint, which results in diameter $d(G)$ of the graph being a lower bound. Thus, for an arbitrary graph G on n vertices:

$$\max\{\lceil \log n \rceil, d(G)\} \leq b(G) \leq n - 1. \tag{1}$$

Given a positive integer n, $B(n)$, called the *broadcast function*, is the minimum number of edges in a broadcast graph on n vertices. A broadcast graph with $B(n)$ edges is called a *minimum broadcast graph* (or *mbg*). The values of $B(n)$ are only known for most $n \leq 32$, $n = 58, 59, 61, 63, 127$ [2, 9, 15, 19, 27, 28, 30, 32, 35, 36]. For larger n, the only known instances of *mbg's* are for n of the form $n = 2^k$ [9], [26],[31] and $n = 2^k - 2$ [4, 24]; for $n = 2^k$ there are three known infinite families of *mbgs*. The obtaining of upper and lower bounds on $B(n)$ is a well-studied problem (see, for example, [1, 8, 14, 16, 17, 19]).

A long-standing conjecture is that $B(n)$ is a non-decreasing function on any interval $[2^{k-1}+1, 2^k]$, for all k [9]. The conjecture is open for all large k. The only partial result appears in [18], where the monotonicity is proved for the first quarter of the interval.

Finding $b(G)$ and also $b(v, G)$ for an arbitrary graph is proved to be NP-complete [13, 33]. The problem remains NP-Hard in more restricted families such as bounded degree graphs [3] and 3-regular planar graphs [29, 23]. Currently the best approximation for this problem is $O(\frac{\log(|V|)}{\log\log(|V|)})b(G)$ [6]. The authors of [5] proved the inapproximability of the problem within a ratio of $3-\epsilon$ for any $\epsilon > 0$. As a recent study of the complexity of the problem, we refer the reader to [10]. Several surveys on broadcasting and related problems can be found in [11, 20, 21, 22].

In this paper, we initiate the study of finding upper on broadcast time $b(G)$ in highly-connected graphs. In particular, we give upper bounds on $b(G)$ for k-connected graphs and graphs with large minimum degree. As far as we know, the problem generally has not been considered from this point of view, perhaps excepting an old result of Frieze and Molloy [12], where it is shown that a sufficiently dense random graph $G(n, p)$ is a broadcast graph. In particular, the authors prove that for the Erdős-Renyi random graph, the threshold of being a broadcast graph essentially matches the threshold of being connected.

The paper is structured as follows. In Section 2, we present an upper bound on broadcast time in k-connected graphs. We bring examples of graphs that match the bound, making it tight, and discuss corollaries tying connectivity to minimum degree. In Section 3, we study the relation between the minimum degree of a graph and its broadcast time. We show the connection using a result on the relationship of the diameter and minimum degree and discuss some extreme examples.

2. Broadcasting in k-connected graphs

In this section, we study the broadcast time of k-connected graphs. A graph G is said to be *k-vertex-connected (k-connected)* if it contains at least $k+1$ vertices, and does not contain a set of $k-1$ vertices whose removal disconnects the graph. The *connectivity* of a graph ($\kappa(G)$) is defined as the largest k such that G is k-connected. Intuitively, graphs of large connectivity should have small broadcast times.

Some of the common graphs are:

- The Path P_n on n vertices has connectivity 1 and broadcast time $n-1$.
- The Star $K_{1,n-1}$ has connectivity 1 and broadcast time $n-1$.
- The Fork/Broom graph, which is a path graph with one of the endpoints being replaced with a star graph has connectivity 1 and broadcast time $n-1$.
- The Cycle C_n is 2-connected and has broadcast time $\lceil \frac{n}{2} \rceil$.
- The Complete graph K_n is $(n-1)$-connected and has broadcast time $\lceil \log n \rceil$.

The following simple result in [25] would be referred to repeatedly.

Proposition 1. *For any complete bipartite graph $K_{k,n-k}$ such that $k \leq n-k$, $b(K_{k,n-k}) = \lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$.*

As we will see, this bound matches the general upper bound for k-connected graphs.

2.1. An upper bound for broadcasting in k-connected graphs

Now, we prove the main result of this section. Its proof uses Kőnig's theorem on the equality of sizes of minimum vertex cover and maximum matching in bipartite graphs.

Theorem 1. *For any k-connected graph G, $b(G) \leq \lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$.*

Proof. Broadcasting in G takes place in three stages. By the end of *Stage 1*, at least k vertices are informed in G. During *Stage 2* at each time unit, k new vertices are being informed. *Stage 3* is the conclusive stage, where less than k vertices are left to be informed.

Let S be the set of currently informed vertices at some time t. Define U to be the set of uninformed vertices (i.e., $U = V - S$). Let the boundary of S, denoted by $\delta(S)$, be the set of uninformed vertices that are adjacent to at least one vertex of S i.e., $\delta(S) = \{v | v \notin S, uv \in E \text{ for some } u \in S\}$.

Claim 1. *Suppose $|S| \geq a$ and $|U| \geq a$ for some $a \leq k$. Then $|\delta(S)| \geq a$.*

Proof of Claim 1. This follows immediately from k-connectivity. Suppose that $|\delta(S)| \leq a - 1 \leq k - 1$. Then $G' = G - \delta(S)$ is connected by definition of k-connectivity. Let $u \in G' \setminus S$. Consider a shortest path P from u to S in G'. Then since P is a shortest path, the second to last vertex of P, say w, is an uninformed vertex adjacent to S, which is a contradiction. $\square$

Claim 2. *Let $G^* = (S, \delta(S))$ be the induced bipartite graph with bipartitions S and $\delta(S)$. Suppose $|S| = a$, and $|\delta(S)| = b$. Then G^* contains a matching of size at least* $\min\{a, b, k\}$.

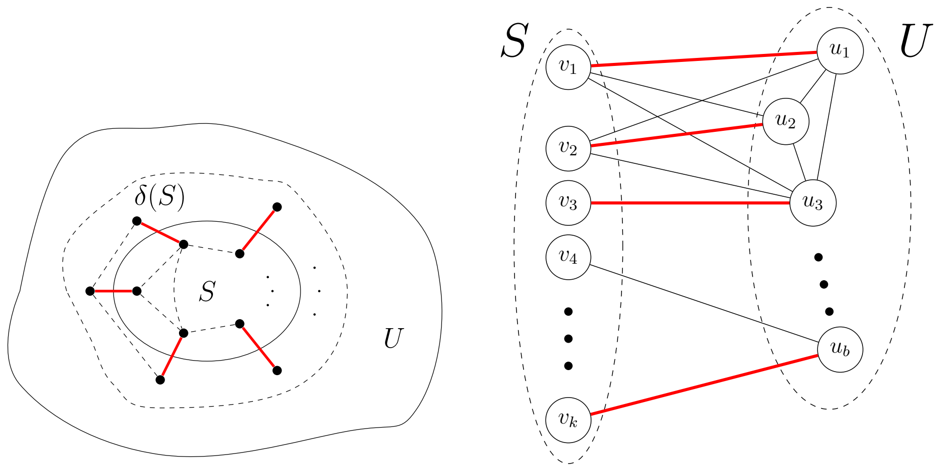
Proof of Claim 2. We will show that $\tau(G^*) \geq \min\{a, b, k\}$, where τ is the size of the minimum vertex cover. Suppose that G^* contains a vertex cover X with $|X| < \min\{a, b, k\}$. Note that $G' = G - X$ is connected by definition, since $\min\{a, b, k\} \leq k$, thus $|X| < k$. Let $S' = S - X$ and $T' = \delta(S) - X$ and note that both are non-empty. Let $x \in T'$ and consider a shortest path P from x to S' in G'. Let $s \in S'$ and w be the last and second to last vertices of P, respectively. By the choice of P, $w \in \delta(S)$, and thus $w \in T'$. But now this implies that X does not cover the edge sw, a contradiction. Thus, $\tau \geq \min\{a, b, k\}$. Recall Kőnig's theorem: In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover. By Kőnig's theorem $\tau \geq \min\{a, b, k\}$ implies that G^* contains a matching of size $\tau \geq \min\{a, b, k\}$. $\square$

Stage 1 of broadcasting takes place in the following way. In the first time unit, $|S| = 1$ and in Claim 1 $a = 1$, thus $\delta(S) \geq 1$, so by Claim 2 there exists a matching M of size $\min\{a, b, k\} = a = 1$. Broadcasting happens through M. At the second time unit, $|S| = 2$, and in Claim 1 $a = 2$, thus $\delta(S) \geq 2$, so by Claim 2 there exists a matching of size $\min\{a, b, k\} = a = 2$.

Similarly, for each time unit $i \leq \lceil \log k \rceil$ there are $|S| = 2^i$ informed vertices and in Claim 1 $a = 2^i$, thus $\delta(S) \geq 2^i$, so by Claim 2 there exists a matching of size $\min\{a, b, k\} = 2^i$. This process continues until $\min\{a, b, k\} \neq a$ anymore, which happens when $a > k$, meaning when $2^j \geq k + 1 \Rightarrow j \geq \lceil \log k \rceil + 1$. At time unit $\lceil \log k \rceil$, *Stage 1* is complete and $a = 2^{\lceil \log k \rceil} \geq k$ vertices are informed.

During *Stage 2* of broadcasting, $|S| = a \geq k$ and we assume that $|U| \geq k$, thus $b \geq k$ (by Claim 1) at some time t. Then at time $t+1$, $|S_{t+1}| \geq |S_t| + k$, since there exists a matching of size at least $\min\{a, b, k\} = k$ by Claim 2. Since k vertices have already been informed in *Stage 1*, there are at most

$n - k$ vertices being informed in *Stage 2* and it takes at most $\lfloor \frac{n-k}{k} \rfloor$ time units.



(a) Stage 2: Matching of size k

(b) Stage 3: Matching of size b

Now, we consider *Stage 3*, i.e. $|U| < k$ and a matching of size $\geq k$ does not exist anymore. $|\delta(S)| = b < k$ and there are less than k uninformed vertices left. Since $a \geq k$ and $b < k$, by Claim 2, there exists a matching of size $\min\{a, b, k\} = b$. *Stage 3* will be complete in a single time unit - informing all vertices in $\delta(S)$ at once.

After completion of *Stage 3*, every vertex in G will be informed and the broadcast will terminate. Now we analyze the total broadcast time.

$k \mid n$: This means, that $n - k$ is also divisible by k, meaning, that all vertices are informed in *Stage 2* and *Stage 3* does not take place. Overall $b(G) \leq \lceil \log k \rceil + \lfloor \frac{n-k}{k} \rfloor = \lceil \log k \rceil + \frac{n}{k} - 1$.

$k \nmid n$: *Stage 3* takes place since there are $1 \leq b < k$ uninformed vertices after Stage 2 is completed. This means the total number of vertices informed during *Stage 2* are $n - k - b$. The overall broadcast time is at most $\lceil \log k \rceil + \lfloor \frac{n-k-b}{k} \rfloor + 1 = \lceil \log k \rceil + \lfloor \frac{n-b}{k} \rfloor - 1 + 1$ and since $1 \leq b < k$, then, $\lfloor \frac{n-b}{k} \rfloor \leq \lceil \frac{n}{k} \rceil - 1$ and thus $b(G) \leq \lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$.

Thus, for any k-connected graph G, $b(G) \leq \lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$. □

Corollary 1. *If a graph G is $\lceil \frac{n}{2} \rceil$-connected, then it is a broadcast graph.*

Proof. Proof is straightforward from the calculations of Theorem 1. For $\lceil \frac{n}{2} \rceil$-connected G,

$$b(G) \leq \left\lceil \log \left\lceil \frac{n}{2} \right\rceil \right\rceil + \left\lceil \frac{n}{\lceil \frac{n}{2} \rceil} \right\rceil - 1 =$$

$$= \begin{cases} \lceil \log n \rceil - 1 + \left\lceil \frac{n}{\frac{n}{2}} \right\rceil - 1 = \lceil \log n \rceil & \text{if } n \text{ is even} \\ \lceil \log (n+1) \rceil - 1 + \left\lceil \frac{n}{\frac{n+1}{2}} \right\rceil - 1 = \lceil \log n \rceil & \text{if } n \text{ is odd} \end{cases}$$

□

Remark 1. *Note that $\lceil \frac{n}{2} \rceil$-connectivity cannot be lowered to $\frac{n}{2} - 1$. Consider the complete bipartite graph $K_{\frac{n}{2}-1, \frac{n}{2}+1}$ when $n = 2^k$ for some $k \in \mathbb{Z}^+$. Proposition 1 implies,*

$$b(K_{\frac{n}{2}-1,\frac{n}{2}+1}) = \left\lceil \log \left(\frac{n}{2} - 1 \right) \right\rceil + \left\lceil \frac{n}{\frac{n}{2} - 1} \right\rceil - 1 =$$

$$= \lceil \log (n-2) \rceil - 1 + \left\lceil \frac{2n}{n-2} \right\rceil - 1 = \log n + 1.$$

Remark 2. *Note that $K_{\frac{n}{2},\frac{n}{2}}$ for even n is $\lceil \frac{n}{2} \rceil$-connected broadcast graph.*

2.2. The bound is tight

Now we present k-connected graphs that attain the bound in Theorem 1 for broadcast time.

General connected graphs

The general upper bound for all graphs in (1) follows from Theorem 1. The path P_n on n vertices has $b(P_n) = n - 1$ matching the upper bound for $k = 1$ in Theorem 1. The star $K_{1,n-1}$ and Fork $F_{n,k}$ graphs also match the bound.

2-connected graphs

The cycle graph on n vertices is 2-connected with $b(C_n) = \lceil \frac{n}{2} \rceil$, which is tight. For the Complete bipartite graph with partitions of size 2 and $n - 2$, $b(K_{2,n-2}) = \lceil \frac{n}{2} \rceil$.

Consider also the Ladder graph L_n on $n = 2m$ vertices. This graph is isomorphic to the Grid $G_{2\times m}$. The broadcast time is $b(G_{2\times m}) = m$, where $m = \frac{n}{2}$.

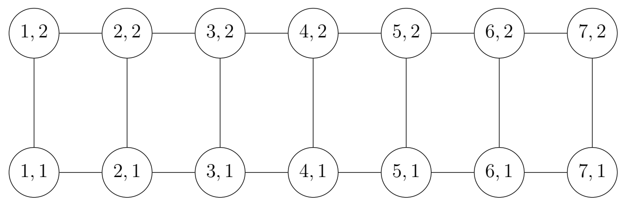


Figure 1: Ladder graph with $m = 7$

There are other examples of 2-connected graphs which attain broadcast time $\lceil \frac{n}{2} \rceil$. For example, see [25].

k-connected graphs

The complete bipartite graph $K_{k,n-k}$, which is a k-connected graph, has broadcast time $b(K_{k,n-k}) = \lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$.

$n/2$ and higher connected graphs

As shown in Corollary 1, all graphs with connectivity at least $\lceil \frac{n}{2} \rceil$ achieve broadcasting in $\lceil \log n \rceil$ time.

2.3. Graphs of large minimum degree

Now we use Theorem 1 to obtain upper bounds on broadcast time of graphs with large minimum degree. We will use the following result.

Theorem 2 ([34]). *Let $1 \leq k \leq n-1$ be an integer. Then every graph G on n vertices with $\delta \geq \left\lceil \frac{n+k-2}{2} \right\rceil$ is k-connected.*

Theorem 3. *Let $1 \leq k \leq n-1$ be an integer. For any graph G on n vertices with $\delta \geq \left\lceil \frac{n+k-2}{2} \right\rceil$, broadcast time $b(G) \leq \lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$.*

Proof. By Theorem 2, the graph G is guaranteed to be k-connected and by Theorem 1, its broadcast time is upper bounded by $\lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$. □

A straightforward series of corollaries follow.

Corollary 2. *Any graph G on n vertices with $\delta \geq \left\lceil \frac{3n}{4} \right\rceil - 1$ is a broadcast graph.*

Proof. We use the fact that $\lceil \frac{n}{4} \rceil = \left\lceil \frac{\lceil \frac{n}{2} \rceil}{2} \right\rceil \forall n \in \mathbb{Z}^+$.

Let G be a graph G on n vertices with $\delta \geq \left\lceil \frac{3n}{4} \right\rceil - 1 = \left\lceil \frac{n}{2} + \frac{n}{4} \right\rceil - 1 = \left\lceil \frac{n + \lceil \frac{n}{2} \rceil - 2}{2} \right\rceil$. By Theorem 2, the graph is $\lceil \frac{n}{2} \rceil$-connected. By Corollary 1, the broadcast time is upper bounded by $\lceil \log n \rceil$ and the graph is a broadcast graph. □

Corollary 3. *Any graph G on n vertices with $\delta \geq \lceil \frac{5n}{8} \rceil - 1$ has broadcast time at most $\lceil \log n \rceil + 1$.*

Proof. Let G be a graph G on n vertices with $\delta \geq \lceil \frac{5n}{8} \rceil - 1 = \lceil \frac{n}{2} + \frac{n}{8} \rceil - 1 = \left\lceil \frac{n + \lceil \frac{n}{4} \rceil - 2}{2} \right\rceil$. By Theorem 2, the graph is $\lceil \frac{n}{4} \rceil$-connected. Thus,

$$b(G) \leq \left\lceil \log \left\lceil \frac{n}{4} \right\rceil \right\rceil + \left\lceil \frac{n}{\lceil \frac{n}{4} \rceil} \right\rceil - 1 \leq \left\lceil \log \left\lceil \frac{n}{4} \right\rceil \right\rceil + 3.$$

If $n = 2^k$, for some positive integer k, then $\left\lceil \log \left\lceil \frac{2^k}{4} \right\rceil \right\rceil + 3 = \lceil \log n \rceil + 1$.

Otherwise if $n = 2^k - x$, for some positive integers k and x, then $\left\lceil \log \left\lceil \frac{2^k - x}{4} \right\rceil \right\rceil + 3 = \lceil \log \lceil 2^{k-2} - \frac{x}{4} \rceil \rceil + 3 \leq \lceil \log \lceil 2^{k-2} - \frac{1}{4} \rceil \rceil + 3 = \lceil \log \lceil 2^{k-2} \rceil \rceil + 3 = k + 1 = \lceil \log(2^k - x) \rceil + 1 = \lceil \log n \rceil + 1$.

Thus, $b(G) \leq \lceil \log n \rceil + 1$. □

Corollary 4. *Any graph G on n vertices with $\delta \geq \lceil \frac{3n}{5} \rceil - 1$ has broadcast time at most $\lceil \log n \rceil + 2$.*

Proof. Let G be a graph G on n vertices with $\delta \geq \lceil \frac{3n}{5} \rceil - 1 = \lceil \frac{n}{2} + \frac{n}{10} \rceil - 1 = \left\lceil \frac{n + \lceil \frac{n}{5} \rceil - 2}{2} \right\rceil$. By Theorem 2, the graph is $\lceil \frac{n}{5} \rceil$-connected.

We use the facts that $\lceil \frac{n}{5} \rceil \leq \lceil \frac{n}{4} \rceil, \forall n \in \mathbb{Z}^+$ and $\left\lceil \log \left\lceil \frac{n}{4} \right\rceil \right\rceil \leq \lceil \log n \rceil - 2$ as seen in Corollary 3.

$$\begin{aligned} b(G) &\leq \left\lceil \log \left\lceil \frac{n}{5} \right\rceil \right\rceil + \left\lceil \frac{n}{\lceil \frac{n}{5} \rceil} \right\rceil - 1 \leq \left\lceil \log \left\lceil \frac{n}{4} \right\rceil \right\rceil + \left\lceil \frac{n}{\lceil \frac{n}{5} \rceil} \right\rceil - 1 \leq \\ &\leq \lceil \log n \rceil - 2 + 5 - 1 = \lceil \log n \rceil + 2. \end{aligned}$$

□

3. Broadcast time and minimum degree

As seen in the previous section, broadcast time for k-connected graphs is upper bounded by $\lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$. We also saw, that since a high minimum degree δ enforces some connectivity on the graph, there is a threshold δ of the graph to obtain a broadcast graph. In this section, we further explore the connection between the minimum degree and broadcast time of a graph.

3.1. Broadcasting in dense graphs

In this section, we are motivated by the following question.

Conjecture 1. *Let $\epsilon > 0$ be a fixed constant and let G be a connected graph on n vertices with minimum degree $\delta(G) \geq \epsilon n$ where n is sufficiently large. Then $b(G) \leq \lceil \log n \rceil + c_\epsilon$, where c_ϵ is some constant that only depends on ϵ.*

Remark 3. *Conjecture 1 is true for $\epsilon > \frac{1}{2}$ by Theorem 3.*

In fact, we believe that when $\epsilon = \frac{1}{2}$, the following is true.

Conjecture 2. *Let G be a graph on n vertices with minimum degree $\delta(G) \geq \frac{n}{2}$, where n is sufficiently large. Then G is a broadcast graph.*

Remark 4. *We note that $\delta \geq \frac{n}{2}$ cannot be relaxed, even if G is connected. Indeed, take an even n and consider two copies of $K_{\frac{n}{2}}$'s and add a single edge between vertices u and v in each copy. The resulting graph has minimum degree $\frac{n}{2} - 1$. Any originator $o \in V \setminus \{u, v\}$ will take at least 2 time units to inform a vertex in the other complete graph. After that broadcasting is complete in $\lceil \log \lceil \frac{n}{2} \rceil \rceil = \lceil \log n \rceil - 1$ time units, resulting in overall $\lceil \log n \rceil + 1$ time.*

We prove a weaker version of the conjecture 1 where $\epsilon = \frac{1}{2}$ and $c_\epsilon = 3$.

Theorem 4. *Let G be a graph on n vertices with $\delta(G) \geq \frac{n}{2}$. Then $b(G) \leq \lceil \log n \rceil + 3$.*

Proof. We consider graphs on $n \geq 8$ vertices since otherwise $\lceil \log n \rceil + 3 \geq n - 1$ (i.e., the upper bound in (1)). We note that G is a simple connected graph and for every pair of non-adjacent vertices u and v the intersection of their neighborhoods (each having cardinality of at least $\frac{n}{2}$) cannot be empty i.e. for $u, v \in V(G), N(u) \cap N(v) \neq \emptyset$. This means, that the diameter of G is at most 2. If G is $\lceil \frac{n}{8} \rceil$-connected, then similar to Stage 1 of Theorem 1, broadcasting in arbitrary greedy manner will double the number of informed vertices at each time unit until time unit $\lceil \log \lceil \frac{n}{2} \rceil \rceil = \lceil \log n \rceil - 1$. There will always be a vertex to inform since $\delta \geq \frac{n}{2}$. After this, the minimum degree does not guarantee doubling of informed vertices, but since the graph is $\lceil \frac{n}{8} \rceil$-connected, there is a matching of size at least $\lceil \frac{n}{8} \rceil$. Informing the remaining at most $n - \lceil \frac{n}{2} \rceil$ vertices by matchings of size $\lceil \frac{n}{8} \rceil$ will take at most 4 time units resulting in $b(G) \leq \lceil \log n \rceil + 3$.

Now assume that G is not $\lceil \frac{n}{8} \rceil$-connected. Then there is a set X with $|X| = \lceil \frac{n}{8} \rceil - 1$ such that $G \setminus X$ is disconnected.

Note that $G \setminus X$ must have exactly two components. Assume it has three or more components. Then, one of the components, say C_3, would have size at most $\frac{n}{3}$, and thus, any vertex in C_3 would have degree in G at most $\frac{n}{3} - 1 + \lceil \frac{n}{8} \rceil - 1 < \frac{n}{2}$, a contradiction.

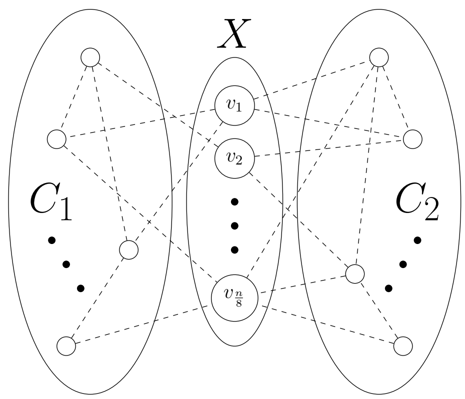


Figure 2: Graph G with $\delta(G) \geq \frac{n}{2}$ with 2 components

Let C_1 and C_2 be the two components of $G \setminus X$. Note that by an identical argument as above, each C_i must contain at least $\frac{n}{2} - \lceil \frac{n}{8} \rceil + 1$ vertices. This now implies, with the condition on the size of X, that in fact for each $i \in \{1, 2\}$, $\frac{n}{2} - \lceil \frac{n}{8} \rceil + 1 \leq |C_i| \leq \frac{n}{2}$.

First, note that independent of the location of the originator, since the diameter is at most 2, at time 2 at least one vertex of C_1 and at least one vertex of C_2 is informed. We now show that we need at most additional $\lceil \log n \rceil$ time such that all vertices of C_1 and C_2 are informed. Note that for each $v \in V(C_1)$, $\deg_{C_1}(v) \geq \frac{n}{2} - \lceil \frac{n}{8} \rceil + 1 = \lfloor \frac{3n}{8} \rfloor + 1$. C_1 is a graph on at most $\frac{n}{2}$ vertices with minimum degree $\delta \geq \lfloor \frac{3n}{8} \rfloor + 1 \geq \frac{\frac{n}{2} + \lfloor \frac{n+8}{4} \rfloor - 2}{2}$. By Theorem 2, C_1 is $\lfloor \frac{n+8}{4} \rfloor$-connected. Since $\frac{n}{4} \leq \lfloor \frac{n+8}{4} \rfloor \leq \frac{n}{2}$ is true $\forall n \in \mathbb{N}, n \geq 6$, the broadcast time of C_1 is upper bounded by:

$$\begin{aligned} b(C_1) &\leq \left\lceil \log \left\lfloor \frac{n+8}{4} \right\rfloor \right\rceil + \left\lceil \frac{n/2}{\lfloor (n+8)/4 \rfloor} \right\rceil - 1 \leq \\ &\leq \left\lceil \log \left(\frac{n}{2} \right) \right\rceil + \left\lceil \frac{n/2}{n/4} \right\rceil - 1 = \\ &= \lceil \log n \rceil - 1 + \left\lceil \frac{2n}{n} \right\rceil - 1 = \lceil \log n \rceil \end{aligned}$$

This shows that C_1 can be informed in $\lceil \log n \rceil$ time units. An identical argument applies for C_2. Thus, in time $\lceil \log n \rceil + 2$ all vertices of C_1 and C_2 are informed.

Now note that for every $v \in X$, there is a set N_v of neighbors of v such that $|N_v| \geq \frac{n}{8}$ and either $N_v \subset C_1$ or $N_v \subset C_2$ (in other words, at least half of the neighbors of v outside X are either in C_1 or C_2). Each such v can choose a distinct informer vertex from its N_v since $|X| \leq \frac{n}{8}$. Thus, it takes one more time to inform X, giving a total of $\lceil \log n \rceil + 3$ time.

Along with $\lceil \frac{n}{8} \rceil$-connected graphs, for all graphs G with $\delta(G) \geq \frac{n}{2}$, $b(G) \leq \lceil \log n \rceil + 3$.

□

3.2. Graphs of small minimum degree

This subsection examines the connection between minimum degree and broadcast time of a graph. Consider the following graph G. Take a path and connect the endpoint vertices to the other neighbor of their neighbors (equivalently, take a path with two pendant K_3's). The minimum degree is 2. The broadcast time of this graph will not be much different from the path. If the originator is one of the vertices in one of the K_3's, then it is easy to see that we need $n - 2$ time units to inform every vertex.

This means that minimum degree alone cannot be a sufficient condition to achieve small broadcast time. Recall that the broadcast time is lower bounded by the diameter of a graph. A classical result of Erdős et al. gives an upper bound on diameter of graphs of minimum degree δ

Theorem 5 (Erdős et al. [7]). *Let G be a connected graph with n vertices and with minimum degree $\delta > 2$. Then diameter $d(G) \leq \left\lfloor \frac{3n}{\delta+1} \right\rfloor - 1$.*

As an example for the tightness of this bound the authors present the following graph. Let $k > 1$, $\delta > 5$, and $V(G) = V_0 \cup V_1 \cup \cdots \cup V_{3k-1}$, where

$$|V_i| = \begin{cases} 1 & \text{if } i \equiv 0 \text{ or } 2 (\text{mod} 3) \\ \delta & \text{if } i = 1 \text{ or } 3k-2 \\ \delta - 1 & \text{otherwise} \end{cases}$$

Let two distinct vertices $v \in V_i, u \in V_j$ be joined by an edge of G if and only if $|i - j| \leq 1$. An instance of such graph is presented in Figure 3.

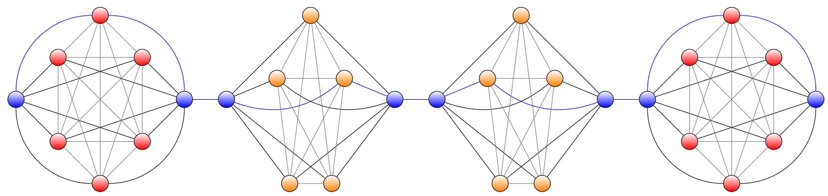


Figure 3: The construction with $n = 30$, $\delta = 6$ and $d(G)11$. A diametral path in blue.

Note that we can broadcast in G in $d(G) + \lceil \log(\delta+2) \rceil - 2$ time units. An easy argument shows that one cannot broadcast faster. We get the $b(G) \geq \left\lfloor \frac{3n}{\delta+1} \right\rfloor + \lceil \log \delta \rceil - 2$.

We believe that this graph of maximal diameter is also the graph with the largest broadcast time. We conjecture the following. We note that we do not even know whether the conjecture is true if the constant 3 is replaced by a larger constant.

Conjecture 3. *If G is a connected graph on n vertices with minimum degree δ, then $b(G) \leq \left\lfloor \frac{3n}{\delta+1} \right\rfloor + \lceil \log \delta \rceil$.*

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